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AN INTEGRALLY CLOSED RING WHICH IS NOT THE INTERSECTION OF VALUATION RINGS

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ABSTRACT. Each commutative ring R which is integrally closed in its total quotient ring $T(R)$ is the intersection of all paravaluation rings of $T(R)$ containing R. In this note an example is given that shows that this statement is not true with "valuation rings" instead of "paravaluation rings". This is an answer of a question asked by J. A. Huckaba in [3].

In this note, all rings are commutative. Every ring has a unit-element, denoted by 1, which is preserved by homomorphisms and inherited by subrings.

Let Γ be a totally ordered commutative group written additively and let $\Gamma_\infty = \Gamma \cup \{\infty\}$ where $\gamma < \infty$, $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$ is defined for all $\gamma \in \Gamma$. If R is a ring then a mapping

$$v\colon R \to \Gamma_\infty, \qquad r \mapsto v(r),$$

is called a paravaluation if the following hold for all x, $y \in R$:

(a) $v(xy) = v(x) + v(y)$.
(b) $v(x+y) \geq \min\{v(x), v(y)\}$.
(c) $v(1) = 0$ and $v(0) = \infty$.

Furthermore, a surjective paravaluation is called a valuation and a subring B of R is called a valuation ring, resp. a paravaluation ring, if there is a valuation, resp. paravaluation, v of R such that $B = \{r \in R \mid v(r) \geq 0\}$. In this case, we write $B = B_v$ and $M_v := \{r \in R \mid v(r) > 0\}$ is a prime ideal of B_v.

Paravaluation rings are a useful tool in commutative ring theory for describing the integral closure of a ring:

Theorem. *Let R be a ring with total quotient ring T. Then R is integrally closed (in T) if and only if R is the intersection of its set of paravaluation rings of T containing R.*

This statement is also valid if T is an arbitrary ring and R a subring of T (cf. [2] and [3, Theorem 9.1]). In his new book, J. A. Huckaba asked whether

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or not "paravaluation" can be replaced by "valuation" in the statement of the Theorem (cf. [3, p. 54 and p. 82]). In this note, we give an example which shows that "paravaluation" cannot be so replaced.

Let K be a commutative field and let $R = K[t]$ be the polynomial ring in the indeterminate t over K.

Proposition 1. *For each $f \in R\backslash K$ there exists a ring S_f such that the following hold*:

(i) *R is a subring of S_f.*
(ii) *There exists a nonzero $z_f \in S_f$ such that $f \cdot z_f = 0$ and $z_f \cdot z_f = 0$.*
(iii) $R \cap Rz_f = \{0\}$.

Proof. Let x be an indeterminate over R and let J be the ideal of $R[x]$ generated by xf and x^2. We define $S := R[x]/J$.

(i) $\Phi: R \to S$, $r \mapsto r + J$. Φ is a monomorphism.
(ii) $z_f := x + J$.
(iii) Let g, $h \in R$ such that $g = hz_f$. We obtain $g - hx \in xfR[x] + x^2R[x]$ and $g \in xR[x]$ follows.

Proposition 2. *There exists a ring S such that the following hold*:

(i) *R is a subring of S.*
(ii) *For each $f \in R\backslash K$ there exists $s_f \in S$, $s_f \neq 0$, such that $f \cdot s_f = 0$ and $s_f \cdot s_f = 0$.*
(iii) *$s_f \cdot s_g = 0$ holds for all f, $g \in R\backslash K$.*
(iv) $R[s_f \mid f \in R\backslash K] = R \oplus (\bigoplus_{f \in R\backslash K} R \cdot s_f)$.

Proof. Let $I = R\backslash K$ be an index set. For each $f \in I$ let S_f be a ring satisfying (i), (ii), (iii) of Proposition 1 and let $S := \prod_{f \in I} S_f$ be the direct product of $\{S_f \mid f \in I\}$.

Since $\Phi: R \to S$, $r \mapsto (r)$ is a monomorphism, (i) is valid. Furthermore, we define for each $f \in R\backslash K$

$$s_f: I \to \bigcup_{i \in I} S_i, \qquad s_f(i) = \begin{cases} z_f & \\ & \text{if} \\ 0 & \end{cases} \begin{matrix} i = f, \\ \\ i \neq f, \end{matrix}$$

and (ii), (iii), (iv) are obvious.

Now, let S be a ring satisfying (i)–(iv) of Proposition 2. We define

$$A := R[s_f \mid f \in R\backslash K].$$

If P denotes the ideal of A generated by all s_f, $f \in R\backslash K$, we get that P is the nilradical of A and P is prime by $A/P \cong R$. Furthermore, each $a \in A$ can be written in one and only one way in the form

$$a = f + p$$

where $f \in R$ and $p \in P$. By construction, we obtain

(*) a $(= f + p$ with $f \in R, p \in P)$ is regular in A if and only if $f \in K^\times$. If a is not regular then $a \cdot p = 0$ holds for a suitable nonzero $p \in P$.

Now, let $v\colon K \to \Gamma_\infty$ be a valuation of K of height 1, i.e. B_v is the only proper subring of K containing B_v. We define

$$B := \{b_0 + b_1 t + \cdots + b_n t^n \mid n \in \mathbf{N},\ b_0 \in B_v,\ b_1, \ldots, b_n \in M_v\}$$

and

$$V := \{f + p \mid f \in B,\ p \in P\}.$$

B is a subring of R and V is a subring of A. By $P \subseteqq V$ and $(*)$, each $f + p \in V$ is regular in V if and only if $f \in B_v$, $f \neq 0$. If $Q(V)$ denotes the total quotient ring of V then

$$\begin{array}{rcl} \pi\colon Q(V) & \to & K[t] \\ (f+p)\cdot(b+q)^{-1} & \mapsto & f \cdot b^{-1} \end{array}$$

is an epimorphism with $\ker \pi = PQ(V)$ where $f \in B$, $b \in B_v$, $b \neq 0$ and p, $q \in P$. Since P is nilpotent, $u(P) = \{\infty\}$ holds for each paravaluation of $Q(V)$. Therefore, π induces a bijection between the set of all valuation rings, resp. paravaluation rings, of $Q(V)$ and the set of all valuation rings, resp. paravaluation rings, of $K[t]$. By $\pi(V) = B$, we are done if we know that B is a paravaluation ring of $K[t]$ which is not equal to the intersection of all valuation rings of $K[t]$ containing B.

(A) B is a paravaluation ring of $K[t]$. We define a valuation w of $K(t)$ such that $B_w \cap K[t] = B$. There exists a totally ordered commutative group Γ' written additively and an element ξ of Γ' such that Γ is a subgroup of Γ' and $0 < n\xi < \gamma$ holds for all $\gamma \in \Gamma$, $\gamma > 0$, and $n \in \mathbf{N}$. By [1, Chap. VI, §10.1 Prop. 1], there exists a unique extension w of v to $K(t)$ with values in Γ'_∞ such that $w(t) = -\xi$. Let $f = k_0 + k_1 t + \cdots + k_n t^n$ and $w(f) = \min\{v(k_0), v(k_1) - \xi, \cdots, v(k_n) - n\xi\}$. We obtain

$$\begin{aligned} w(f) \geq 0 &\Leftrightarrow v(k_0),\ v(k_1) - \xi, \cdots, v(k_n) - n\xi \geq 0 \\ &\Leftrightarrow v(k_0) \geq 0,\ v(k_1) \geq \xi, \cdots, v(k_n) \geq n\xi \\ &\Leftrightarrow v(k_0) \geq 0,\ v(k_1) > 0, \cdots, v(k_n) > 0 \\ &\Leftrightarrow f \in B. \end{aligned}$$

(B) $B_v[t]$ is the intersection of all valuation rings of $K[t]$ containing B. First of all, $B_v[t]$ is a valuation ring of $K[t]$ since

$$\begin{array}{rcl} u\colon K[t] & \longrightarrow & \Gamma_\infty \\ k_0 + \cdots + k_n t^n & \to & \min\{v(k_0), \cdots, v(k_n)\} \end{array}$$

is a valuation of $K[t]$ such that $B_u = B_v[t]$.

Now, let u be an arbitrary valuation of $K[t]$ such that $B \subseteqq B_u$. If $K \subseteqq B_u$ then $B_u = K[t]$ follows by $B \subseteqq B_u$. Thus, let $K \not\subseteqq B_u$. Since v has height

1, we obtain $B_u \cap K = B_v$. Clearly, $u(t) \geq 0$ implies $B_v[t] \subsetneqq B_u$. Thus, let $u(t) < 0$. Then, $u(t^n) < u(k) < 0$ cannot hold for any $k \in K$, $n \in \mathbf{N}$. Otherwise we obtain $u(k^{-1}t^n) < 0$ for at least one $k^{-1} \in M_v$ and one $n \in \mathbf{N}$, i.e. $k^{-1}t^n \notin B_u$ but $k^{-1}t^n \in B$. Therefore, $u(k) < u(t^n) < 0$ is valid for all $k \in K \backslash B_v$ and $n \in \mathbf{N}$. Thus, $nu(t) \notin \Gamma$ holds for all $n \in \mathbf{N}$. By [1, Chap. VI, §10.1 Prop.1],

$$u(k_0 + k_1 t + \cdots + k_n t^n) = \min\{v(k_0),\ v(k_1) + u(t),\ \cdots, v(k_n) + nu(t)\}$$

follows for all $k_0, k_1, \cdots, k_n \in K$. Since $v(k)+nu(t) \neq 0$ holds for all $k \in K$, $n \in \mathbf{N}$, no $f \in K[t]$ satisfies $u(tf) = 0$, i.e. u is no valuation.

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