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THE HENSELIAN DEFECT FOR VALUED FUNCTION FIELDS

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ABSTRACT. The notion of defect for finite algebraic extensions of valued fields is classical and due to Ostrowski. Recently Matignon has generalized Ostrowski's definition to rk 1 (residually transcendental) valued function fields and used it to prove a very sharp version of the genus reduction inequality for 1-dim function fields. The further generalization of the notion of defect to valued function fields of arbitrary rk is treated here.

Let $(K/K_0, v)$ be a valued function field of $\dim n$, i.e., K/K_0 is a finitely generated field extension of deg of transcendence n and v is a valuation of K. Let $V_0 \subset V$, $k_0 \subset k$, and $G_0 \subset G$ be the respective valuation rings, residue fields, and value groups of the extension $K_0 \subset K$; and let $*$ denote image under the v-residue map $V \to V/m_V = k$.

A transcendence basis $t = \{t_1, \dots, t_n\}$ of K/K_0 will be called a *residually transcendental* (abbreviated tr.) *basis* of the valued function field if $v(t_i) \geq 0$ $(i = 1, \dots, n)$ and the set of v-residues $t^* = \{t_1^*, \dots, t_n^*\}$ is algebraically independent over k_0. The function field will be called *residually tr.* if there exists a residually tr. basis.

A transcendence basis t is residually tr. iff $v \mid K_0(t)$ is the *inf extension*, denoted v_0^t, of v_0 w.r.t. t, i.e., iff for all $f(t)$ in $K_0[t], v(f) =$ the inf of the values of the coefficients of f. Note that the value group of v_0^t is clearly G_0, and the residue field is $k_0(t^*)$. (Cf. [3, p. 161, Proposition 2].)

If t is a residually tr. basis, the *henselian defect at* t is defined to be

$$D^h(t) := [K^h : K_0(t)^h]/IR,$$

where K^h denotes henselization, $I = [G: G_0]$, and $R = [k: k_0(t^*)]$.

We shall prove here the

0. **Independence Theorem.** *Let* $(K/K_0, v)$ *be a residually tr. valued function field. Then* $D^h(t)$ *is independent of the choice of residually tr. basis* t.

The case that K/K_0 is simple tr. has been proved in [13, Theorem 2.2]. Also, Matignon [10, p. 191, Corollary 1] has proved that if $\text{rk}\, v = 1$, then the

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completion defect $D^\frown(t)$ is independent of t, where $D^\frown(t)$ is defined analogously to $D^h(t)$ by using the completion instead of the henselization; this result, which plays a key role in the proof of Matignon's genus reduction inequality [10], follows readily from Theorem 0; see 4.2. Finally, we should mention that F.-V. Kuhlmann has independently proved the Independence Theorem; see 4.3.

1. Preliminaries

Fix throughout §1 *a finite algebraic extension of valued fields* $(L_0, w_0) \subset (L, w)$, *with value groups* $H_0 \subset H$ *and residue fields* $1_0 \subset 1$.

We use L^h to denote henselization and $L^\frown$ to denote completion. Recall that both L^h and $L^\frown$ have the same residue field and value group as L and that any finite generating set for L/L_0 is also a generating set for L^h/L_0^h and for $L^\frown/L_0^\frown$; cf. [1, pp. 175–179] or [4, p. 131] for henselizations and [17, pp. 45–47, §§4 and 5] or [3, p. 121] for completions. We shall call $[L^h : L_0^h]$ the *henselian deg* of w/w_0.

A set of extensions $w_1 = w, w_2, \dots, w_n$ of w_0 to L is called a *a complete set of extensions of* w_0 *to* L if every extension of w_0 to L is equivalent to one of these and no two of these are equivalent.

1.1. The fundamental equality.

$$[L: L_0] = \sum_{i=1}^{n} [(L, w_i)^h : (L_0, w_0)^h]$$

(Apply [4, p. 125, (17.3)] and the fact that, in the terminology of that reference, L_0^h/L_0 is a separable, "allowable" extension.) The sum is taken over a complete set of extensions $w_1 = w, \cdots, w_n$ of w_0 to L. In words, the fundamental equality says that the deg is the sum of the Henselian degs.

1.2. **Definition of the defects.** Let $e = [H: H_0]$ and $f = [1: 1_0]$; that is, e is the index and f is the residue deg. We define two notions of defect for the extension $(L_0, w_0) \subset (L, w)$, the *henselian defect* and the *completion defect*, respectively, as follows:

(i) $\operatorname{def}^h(w/w_0) = [(L, w)^h : (L_0, w_0)^h]/ef$
(ii) $\operatorname{def}^\frown(w/w_0) = [(L, w)^\frown : (L_0, w_0)^\frown]/ef$.

We can now restate the fundamental equality 1.1:

$$[L: L_0] = \sum_{i=1}^{n} \operatorname{def}^h(w_i/w_0) e_i f_i . \tag{1.1'}$$

Note that each defect is a rational number ≥ 1. Much of the usefulness of these notions is due to the classical

Theorem. *Let* $p = \operatorname{char} l_0$ *if* $\operatorname{char} l_0 > 0$ *and* $p = 1$ *if* $\operatorname{char} l_0 = 0$.

(i) (*Ostrowski*) *If* rk $w_0 = 1$, *then* $\operatorname{def}^\frown(w/w_0) = p^i$ *for some* $i \geq 0$; *and if* w_0 *is discrete* rk 1, *then* $\operatorname{def}^\frown(w/w_0) = 1$ (*cf.* [16, p. 355] *and* [3, p. 148, Corollary 2]).

(ii) (*E. Artin-Ostrowski*) *If* $\operatorname{rk} w_0$ *is arbitrary, then* $\operatorname{def}^h(w/w_0) = p^i$ *for some* $i \geq 0$ (*cf.* [1, p. 180, Proposition 15], *or* [3, p. 190, Exercise 9]).

1.3. **Comparison of the defects.**

Proposition ([13, §1.3]). $\operatorname{def}^h(w/w_0) \leq \operatorname{def}^\frown(w/w_0) Q^\frown(w/w_0)$, *and* $=$ *holds if* $(L_0, w_0)^\frown$ *is henselian* (*in particular,* $=$ *holds if* $\operatorname{rk} w_0 = 1$). *Here* $Q^\frown(w/w_0)$ *is called the* inseparability quotient *and is defined by*

$$Q^\frown(w/w_0) = [L: L_0]_{\text{ins}} / [(L, w)^\frown : (L_0, w_0)^\frown]_{\text{ins}}$$

where $[\ ]_{\text{ins}}$ *denotes deg of inseparability.*

Note that $Q^\frown(w/w_0) = 1$ whenever $L_0^\frown / L_0$ or L/L_0 is separable, and in general equals p^i for some $i \geq 0$, where $p = \operatorname{char} L_0$ (apply [19, p. 119, Corollary 2 and p. 114, Lemma 1]). For a discrete rk 1 example with $[L: L_0] = Q^\frown(w/w_0) = p > 0$, cf. [20, p. 62]; this example has $\operatorname{def}^\frown = 1$ and $\operatorname{def}^h = p$. On the other hand, there exist examples with $\operatorname{def}^h = 1$ and $\operatorname{def}^\frown$ an arbitrary rational number ≥ 1; cf. [13, Example 2.5].

1.4 **Stability (cf.** [2, p. 160, Proposition 6], [7, p. 57]). The valued field (L_0, w_0) is called *stable* if for every finite algebraic extension (L, w) of (L_0, w_0), $\operatorname{def}^h(w/w_0) = 1$.

2. Proof of the theorem for K_0 algebraically closed

The terminology of the introduction will be in effect.

2.1. **Theorem.** *Let* $(K/K_0, v)$ *be a residually* tr. *valued function field, and let* $t = \{t_1, \ldots, t_n\}$ *be a residually* tr. *basis. If* K_0 *is algebraically closed, then* $\operatorname{def}^h(K/K_0(t)) = 1$.

Proof. Let $v_0 = v \mid K_0$ and $v_0^t = v \mid K_0(t)$. Note that $\operatorname{rk} v_0 = \operatorname{rk} v_0^t = \operatorname{rk} v$.
Case (i). $\operatorname{rk} v = 1$. We work inside $K^\frown$. K_0 is algebraically closed $\Rightarrow K_0^\frown$ is algebraically closed ([2, p. 146, Proposition 3] $\Rightarrow K_0^\frown(t)$ is stable ([2, p. 215, Proposition 3]) $\Rightarrow \operatorname{def}^h(K_0^\frown K / K_0^\frown(t)) = 1$, by 1.4.

But by 1.3, in rk 1

$$\operatorname{def}^h(K_0^\frown K / K_0^\frown(t)) = \operatorname{def}^\frown(K_0^\frown K / K_0^\frown(t)) Q^\frown(K_0^\frown K / K_0^\frown(t)), \tag{2.2}$$

so $\operatorname{def}^\frown(K_0^\frown K / K_0^\frown(t)) = 1$ and $Q^\frown(K_0^\frown K / K_0^\frown(t)) = 1$.
Since $(K_0^\frown K)^\frown = K^\frown$ and $(K_0^\frown(t))^\frown = K_0(t)^\frown$, we have by definition of $\operatorname{def}^\frown$, $\operatorname{def}^\frown(K/K_0(t)) = 1$. It remains to show $Q^\frown(K/K_0(t)) = 1$, for then another application of 1.3 yields the desired result. Since we know $Q^\frown(K_0^\frown K / K_0^\frown(t)) = 1$, it suffices to prove the

Claim. $[K: K_0(t)]_{\text{ins}} = [K_0^\frown K : K_0^\frown(t)]_{\text{ins}}$. Since t is a residually tr. basis, the set t remains algebraically independent over $K_0^\frown$. Therefore K and $K_0^\frown$ are algebraically independent over K_0; and since K_0 is algebraically closed, then

K and $K_0^{\frown}$ are linearly disjoint over K_0 (cf. [18, p. 18, Theorem 5]). The claim follows by [19, p. 114, Lemma 1]. Q.E.D. for Case (i).

Case (ii). $\operatorname{rk} v$ is finite. We proceed by induction on $\operatorname{rk} v$. We shall prove in Case (ii) the following equivalent form of Theorem 2.1.

2.1′ **Theorem.** *Let K_0 be an algebraically closed field, let t be a finite set of indeterminates, and let K be a finite algebraic extension of $K_0(t)$. Let v_0 be a valuation of K_0, v_0^t be the inf extension of v_0 w.r.t. t, and $v_1, \ldots, v_m$ be a complete set of extensions of v_0^t to K. Then*

$$[K : K_0(t)] = R(v_1/v_0^t) + \cdots + R(v_m/v_0^t),$$

where $R(v_i/v_0^t) =$ residue deg of v_i/v_0^t.

Assume $\operatorname{rk} v_0$ is finite, > 1.

For any finite rk valuation of a field, there is a unique (up to equivalence) rk 1 valuation of the same field whose valuation ring contains the valuation ring of the given valuation. Let $w_0, w_1, \ldots, w_m$ be the rk 1 valuations corresponding in this way to $v_0, v_1, \ldots, v_m$. Some of $w_1, \ldots, w_m$ may be equivalent (i.e., have the same valuation ring), so let us restrict ourselves to a complete subset of inequivalent valuations, say $w_1, \ldots, w_s$. If w_0^t denotes the inf extension of w_0 w.r.t. t, it follows that w_0^t is the rk 1 valuation corresponding to v_0^t.

The rk 1 w-valuations satisfy the hypothesis of 2.1′, and therefore by the previously proved rk 1 case,

$$[K : K_0(t)] = R(w_1/w_0^t) + \cdots + R(w_s/w_0^t), \tag{2.3}$$

where $R(\ )$ denotes residue deg.

Now let w be any one of the valuations $w_1, \ldots, w_s$, and let, say, $v_1, \ldots, v_q$ be those elements of $\{v_1, \ldots, v_m\}$ whose valuation rings are contained in the valuation ring of w (i.e., $v_1, \ldots, v_q$ are those elements of $\{v_1, \ldots, v_m\}$ which are in the dependence class determined by w). Let $l_0 =$ residue field of w_0, $l_w =$ residue field of w; and denote image under the w-residue map by (a bar). Then l_w is finite algebraic over $l_0(\bar{t})$ and the set $\bar{t}$ is algebraically independent over l_0. Thus, we have valuations $\bar{v}_0$ of l_0, $(v_0^t)^- = \bar{v}_0^{\bar{t}}$ of $l_0(\bar{t})$, and $\bar{v}_1, \ldots, \bar{v}_q$ of l_w (where a valuation $\bar{v}$ corresponds to a valuation v by applying the w-residue map to the valuation ring of v).

$$\begin{array}{ccccc}
K_0 & \text{——} & K_0(t) & \text{——} & K \\
{\scriptstyle w_0}\big\downarrow & & {\scriptstyle w_0^t}\big\downarrow & & {\scriptstyle w}\big\downarrow \\
l_0 & \text{——} & l_0(\bar{t}) & \text{——} & l_w \\
\big\downarrow{\scriptstyle \bar{v}_0} & & \big\downarrow{\scriptstyle \bar{v}_0^{\bar{t}}} & & \big\downarrow{\scriptstyle \bar{v}} \\
k_0 & \text{——} & k_0(t^*) & \text{——} & k_v
\end{array}$$

By induction hypothesis,

$$R(w/w_0^t) = [l_w : l_0(\bar{t})] = R(\bar{v}_1/\bar{v}_0^{\bar{t}}) + \cdots + R(\bar{v}_q/\bar{v}_0^{\bar{t}}). \tag{2.4}$$

Since $R(v_i/v_0^t) = R(\bar{v}_i/\bar{v}_0^{\bar{t}})$, we can substitute the expressions (2.4) into (2.3) to obtain the desired equality. Q.E.D. for Case (ii).

Case (iii). $\operatorname{rk} v_0$ arbitrary. Our approach will be to drop down to a suitable extension $K_0' \subset K'$, where K_0' is the algebraic closure of a finitely generated extension of the prime field. This forces $\operatorname{rk}(v \mid K')$ to be finite, and we can then apply Case (ii).

First we need a lemma.

2.5 **Lemma.** *Let $(K/K_0, v)$ be a residually tr. valued function field, let t be a residually tr. basis, and let $K = K_0(t, a)$, where a denotes a finite generating set for $K/K_0(t)$.*

Then there exists a finite subset S of K_0 such that if K_0' is any subfield of K_0 containing S and $K' = K_0'(t, a)$, then

(i) *$K'/K_0'(t)$ is algebraic,*
(ii) *$[K^h : K_0(t)^h] = [K'^h : K_0'(t)^h]$, and*
(iii) residue deg $(K/K_0(t)) \le$ *residue deg* $(K'/K_0'(t))$.

Moreover, if K_0' is chosen to be algebraically closed, then $=$ holds in (iii).

(For ease of notation, let E^h, E'^h denote the left and right sides of (ii) and R, R' the left and right sides of (iii)).

Before proving 2.5, we shall finish the proof of Case (iii). Let S be given by 2.5; let K_0' be the algebraic closure of $P(S)$, where P is the prime subfield of K_0; and let $K' = K_0'(t, a)$, where a is a generating set for $K/K_0(t)$, as in 2.5. Note that $v \mid K_0'$ has finite rk (cf. [20, p. 8, Theorem 3]). Also, since K_0' and K_0 are algebraically closed, both $K/K_0(t)$ and $K'/K_0'(t)$ have index 1. By definition of def^h, $R\operatorname{def}^h(K/K_0(t)) = E^h$ and $R' \operatorname{def}^h(K'/K_0'(t)) = E'^h$; by 2.5–(ii) $E^h = E'^h$; and by the final assertion of 2.5, $R = R'$. Thus, $\operatorname{def}^h(K/K_0(t)) = \operatorname{def}^h(K'/K_0'(t))$. But by Case (ii) $\operatorname{def}^h(K'/K_0'(t)) = 1$. This concludes the proof of 2.1.

Proof of Lemma 2.5. Let $v_0 = v \mid K_0$ and $v_0^t = v \mid K_0(t)$; let $v_1 = v, \ldots, v_m$ be a complete set of extensions of v_0^t to K; let k_i = residue field of v_i $(i = 0, \ldots, m)$; and write $k_i = k_0(t^*, z^{(i)})$, where $z^{(i)}$ denotes a finite set of elements and t^* denotes the set of v_0^t-residues of the elements of t.

By enlarging the set a if necessary, we may assume a contains a set of elements having v_i-residue $z^{(i)}$ $(i = 1, \ldots, m)$.

Let F be the finite subset of $K_0(t)$ consisting of the union of the following three sets:

F_1. Choose a finite basis in $K_0(t)[X]$ for the ideal of a over $K_0(t)$, and let F_1 be the set of $K_0(t)$-coefficients appearing in the elements of this basis.

F_2. For each i in $\{1, \ldots, m\}$ choose a finite basis in $k_0(t^*)[X]$ for the ideal of $z^{(i)}$ over $k_0(t^*)$. For each $k_0(t^*)$-coefficient c^* appearing in these basis elements select an element c in $K_0(t)$ having c^* as its v_0^t-residue, and let F_2 be the resulting set of c's.

F_3. For each $i \neq j$ in $\{1, \ldots, m\}$, choose an element b_{ij} in $K = K_0(t)[a]$ such that $v_i(b_{ij}) \geq 0$ and $v_j(b_{ij}) < 0$. Write each b_{ij} as an element of $K_0(t)[a]$, and let F_3 be the set of $K_0(t)$-coefficients appearing in the resulting expressions for the b_{ij}.

Finally, write each element of F as an element of $K_0(t)$, and let S be the set of K_0-coefficients appearing in the resulting expressions.

Now let K_0' be any subfield of K_0 containing S; let $K' = K_0'(t, a)$; and let $v_0' = v \mid K_0', v_0'^t = v \mid K_0'(t)$, and $v_i' = v_i \mid K'$ $(i = 1, \ldots, m)$. Note that $F \subset K_0'(t)$; and, in particular, $F_3 \subset K_0'(t)$ implies that $v_1', \ldots, v_m'$ are inequivalent, and $F_1 \subset K_0'(t)$ implies 2.5–(i) (cf. [18, p. 15, Theorem 3]).

Fix a v_i, and work inside a henselization $(K, v_i)^h$. Then $K^h = K_0(t)^h(a)$ and $K'^h = K_0'(t)^h(a)$, so $K_0'(t)^h \subset K_0(t)^h$ implies

$$[K^h : K_0(t)^h] \leq [K'^h : K_0'(t)^h]. \tag{2.6}$$

For ease of notation, let E_i^h and $E_i'^h$ denote the left and right sides of (2.6), so that (2.6) reads $E_i^h \leq E_i'^h$.

Since $F_1 \subset K_0'(t)$, we have (cf. [18, p. 15, Theorem 3]) $K_0(t)$ and $K_0'(t)(a)$ are linearly disjoint over $K_0'(t)$. Therefore

$$[K_0(t)(a) : K_0(t)] = [K_0'(t)(a) : K_0'(t)], \tag{2.7}$$

or, in abbreviated notation, $E = E'$.

Putting together 1.1, (2.6), and (2.7), we have

$$\begin{array}{ccccccccc} E & = & E_1^h & + \cdots + & E_m^h & & \\ \| & & \wedge| & & \wedge| & & \\ E' & = & \hat{E}_1'^h & + \cdots + & \hat{E}_m'^h & + \cdots + & E_n'^h, \end{array}$$

where the E_i^h (resp. the $E_j'^h$) are the henselian degrees of a complete set of extensions of v_0^t (resp. $v_0'^t$) to K (resp. K').

It follows from (2.8) that $m = n$ and that the inequalities of (2.8) are equalities. In particular, since $v = v_1$, the equality $E_1^h = E_1'^h$ yields 2.5–(ii).

Now let k_i' = residue field of $v_i', i = 0, \ldots, m$. Note that by our intial enlargement of a, $k_0'(t^*, z^{(i)}) \subset k_i'$ $(i = 1, \ldots, m)$; and therefore

$$[k_i' : k_0'(t^*)] \geq [k_0'(t^*)(z^{(i)}) : k_0'(t^*)] \qquad (i = 1, \ldots, m). \tag{2.9}$$

Since $F_2 \subset K_0'(t)$, we have (cf. [18, p. 15, Theorem 3]) $k_0(t^*)$ and $k_0'(t^*)(z^{(i)})$ are linearly disjoint over $k_0'(t^*)$; and therefore

$$[k_0'(t^*)(z^{(i)}) : k_0'(t^*)] = [k_0(t^*)(z^{(i)}) : k_0(t^*)] \qquad (i = 1, \ldots, m). \tag{2.10}$$

Then (2.9) and (2.10) yield 2.5–(iii).

Finally, assume K_0' is algebraically closed. Then k_0' is also algebraically closed, and therefore by [9, p. 58, Theorem 4], $k_0'(t^*, z^{(i)})$ is algebraically closed in $k_0(t^*, z^{(i)})$. But

$$k_0'(t^*, z^{(i)}) \subset k_i' \subset k_0(t^*, z^{(i)})$$

and k_i' is algebraic over $k_0'(t^*, z^{(i)})$; so then $k_0'(t^*, z^{(i)}) = k_i'$. Therefore (2.10) yields the final assertion of the lemma. □

3. Proof of the theorem

3.1 Theorem (Independence of def^h for residually tr. function fields). *Let $(K/K_0, v)$ be a residually tr. valued function field. Then for any two residually tr. bases $t^{(1)}, t^{(2)}$ of the function field,* $\operatorname{def}^h(K/K_0(t^{(1)})) = \operatorname{def}^h(K/K_0(t^{(2)}))$.

First we need an addition to Lemma 2.5.

2.5′ Lemma. *Assume the hypotheses and notation of* 2.5. *If K_0 is algebraically closed, then*

(iv) $\operatorname{index}(K'/K_0'(t)) = \operatorname{def}^h(K'/K_0'(t)) = 1$.

Proof. Let $I' = \operatorname{index}(K'/K_0'(t))$. By 2.5–(ii) and (iii) and the definition of def^h, we have (in the notation of 2.5)

$$\mathrm{R}\operatorname{def}^h(K/K_0(t)) = E^h = E'^h = \mathrm{I}'\mathrm{R}' \operatorname{def}^h(K'/K_0'(t)) \geq R' \geq R. \tag{3.2}$$

But $\operatorname{def}^h(K/K_0(t)) = 1$ by 2.1; so the inequalities of (3.2) must actually be equalities. Then $\mathrm{I}'\mathrm{R}' \operatorname{def}^h(K'/K_0'(t)) = R'$, which implies $\mathrm{I}' \operatorname{def}^h(K'/K_0'(t)) = 1$. □

3.3 *Proof of* 3.1. Let K_0^{alg} be the algebraic closure of K_0, and fix an extension (again denoted v) of v to $K_0^{\text{alg}}K$. By 2.5′ applied to $K_0^{\text{alg}} \subset K_0^{\text{alg}}K$, there exists a finite algebraic extension K_0' of K_0 such that $\operatorname{def}^h(K'/K_0'(t^{(1)})) = \operatorname{def}^h(K'/K_0'(t^{(2)})) = 1$, where $K' = K_0'K$.

$$\begin{array}{ccc} K_0'(t^{(i)}) & \text{——} & K' = K_0'K \\ | & & | \\ K_0(t^{(i)}) & \text{——} & K \end{array}$$

Since $\operatorname{def}^h(K'/K_0'(t^{(i)}))\operatorname{def}^h(K_0'(t^{(i)})/K_0(t^{(i)})) = \operatorname{def}^h(K'/K)\operatorname{def}^h(K/K_0(t^{(i)}))$, it only remains to note that $\operatorname{def}^h(K_0'(t^{(1)})/K_0(t^{(1)})) = \operatorname{def}^h(K_0'(t^{(2)})/K_0(t^{(2)}))$, which is a consequence of the observation that the extensions $K_0(t^{(1)}) \subset K_0'(t^{(1)})$ and $K_0(t^{(2)}) \subset K_0'(t^{(2)})$ are isomorphic as valued field extensions. □

4. Remarks

4.1. Let (K_0, v_0) be a valued field, and let v_0^t be the inf extension of v_0 w.r.t. a set of n indeterminates $t = \{t_1, \dots, t_n\}$.

Theorem 2.1 can be rephrased as follows:

Theorem. *If K_0 is algebraically closed, then $(K_0(t), v_0^t)$ is stable.*

In the same vein, the argument of 3.3 yields the more general

Theorem. *If (K_0, v_0) is stable, then $(K_0(t), v_0^t)$ is stable.*

Proof. Proceed as in 3.3, except that to see $\operatorname{def}^h(K_0'(t)/K_0(t)) = 1$ in 3.3, use the

Lemma. *Let $(K_0, v_0) \subset (K_1, v_1)$ be an extension of valued fields with K_1/K_0 finite algebraic; let $t = \{t_1, \dots, t_n\}$ be a set of n indeterminates; and let v_0^t, v_1^t be the* inf *extensions of v_0, v_1 resp. w.r.t. t. Then $\operatorname{def}^h(v_1/v_0) = \operatorname{def}^h(v_1^t/v_0^t)$.*

Proof (Matignon). Since v_1/v_0 and v_1^t/v_0^t have the same index and residue deg, the desired equality is equivalent to $[(K_1, v_1)^h : (K_0, v_0)^h] = [(K_1(t), v_1^t)^h : (K_0(t), v_0^t)^h]$.

Let $v_1, v_2, \dots, v_n$ be a complete set of extensions of v_0 to K_1. Then the corresponding inf extensions $v_1^t, v_2^t, \dots, v_n^t$ form a complete set of extensions of v_0^t to $K_1(t)$. Therefore, by 1.1,

$$[K_1 : K_0] = \sum_{i=1}^{n} [(K_1, v_i)^h : (K_0, v_0)^h]$$

$$\|$$

$$[K_1(t) : K_0(t)] = \sum_{i=1}^{n} [(K_1(t), v_i^t)^h : (K_0(t), v_0^t)^h].$$

But $[(K_1, v_i)^h : (K_0, v_0)^h] \geq [(K_1(t), v_i^t)^h : (K_0(t), v_0^t)^h]$ $(i = 1, \dots, n)$, since any finite generating set for K_1/K_0 is also a generating set for K_1^h/K_0^h and $K_1(t)^h/K_0(t)^h$; so it follows that these inequalities must actually be equalities. □

The rk 1 case of the first theorem is due to Grauert-Remmert [5, p. 119] and that of the second theorem to Gruson [7, p. 66, Theorem 3]; cf. [2, pp. 214–220] for "a simplified version of Gruson's approach."

4.2. Matignon [10, p. 191, Corollary 1] has proved that $\operatorname{def}^\frown(K/K_0(t))$ is independent of t in the rk 1 case. (A proof of this result in the simple tr. case is given in [12, Theorem 2.5]. Note also that the rk 1 hypothesis is necessary by [12, Example 2.6].) This result is equivalent to the rk 1 case of our independence theorem; for in rk 1, by 1.3,

$$\operatorname{def}^h(K/K_0(t)) = \operatorname{def}^\frown(K/K_0(t))\, Q^\frown(K/K_0(t)),$$

and by [13, Remark 2.4.6] $Q^{\wedge}(K/K_0(t))$ is independent of t (for any residually tr. valued function field).

4.3. Recent interest in the notion of defect has been stimulated by the key role it plays in the proof of Matignon's remarkable genus reduction inequality [10] (for 1-dim rk 1 valued function fields), and by efforts to generalize this result to valuations of arbitrary rk; cf. [6], [8], [11]. In addition, in [12] and [13] the defect supplied the missing ingredient needed for the proof of the conjectures of [14] and [15] concerning the structure of simple tr. extensions of valued fields.

The proof of the independence theorem given here was inspired, in bare outline, by the proof of the simple tr. case given in [13, Theorem 2.2–(i)] and [12, Theorem 2.5]. The technique of reducing to rk 1 is classical, but it was brought to my attention by a letter (October 1987) from Barry Green and F. Pop to Matignon, in which they pointed out how it could be used to remove the rk 1 restriction from another of Matignon's results. Recently it was also brought to my attention that F.-V. Kuhlmann of Heidelberg had already used and proved generalizations of the principal results given here in his work towards his thesis.

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