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GROUP C^*-ALGEBRAS AS ALGEBRAS OF "CONTINUOUS FUNCTIONS" WITH NON-COMMUTING VARIABLES

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ABSTRACT. It is shown that a system of commutation relations depending on the structure constants of a Lie algebra $\mathfrak{g}$ leads to a C^*-algebra which is isomorphic to the group C^*-algebra of the simply connected Lie group associated with $\mathfrak{g}$.

INTRODUCTION

Let $\mathbf{G}$ be a connected simply connected Lie group, $\mathfrak{g}$ be the corresponding Lie algebra (of the right invariant vector fields), $X_1, \dots, X_N$ form a basis in $\mathfrak{g}$ such that

$$[X_m, X_n] = \sum_{k=1}^{N} c_{mn}^{k} X_k.$$

If U is a strongly continuous unitary representation of $\mathbf{G}$ acting on $\mathscr{H}$ and dU the infinitesimal representation associated with U, $A_j = \overline{dU(X_j)}$, then we define operators B_j for $j = 0, 1, \dots, N$ as

$$B_0 = \left(\overline{1 + \sum_{k=1}^{N} A_k^* A_k}\right)^{-1},$$
$$B_j = A_j B_0.$$

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It is easy to establish that B_j are bounded operators and satisfy the following conditions:

$$B_0^* = B_0, \tag{1}$$

$$B_m^* = -B_0 B_m, \qquad m = 1, \dots, N, \tag{2}$$

$$B_0(1 - B_0) = \sum_{k=1}^{N} B_k^* B_k, \tag{3}$$

$$B_n^* B_m - B_m^* B_n = \sum_{k=1}^{N} c_{mn}^k B_0 B_k, \qquad m, n = 1, \dots, N, \tag{4}$$

$$B_m^* + B_m = \sum_{k,r=1}^{N} c_{mk}^r (B_r^* B_k + B_k^* B_r), \qquad m = 1, \dots, N. \tag{5}$$

The second formula is a consequence of antisymmetry of A_m, the fourth we get considering the commutator $[X_m, X_n]$ and the fifth similarly computing $[X_m, \Delta]$ $(\Delta = \sum_{k=1}^{N} X_k^2$ in the enveloping algebra).

Starting from the above relations we shall construct a C^*-algebra which will appear to be isomorphic with $C^*(\mathbf{G})$.

I take this opportunity to thank Prof. S. L. Woronowicz who suggested the problem and gave much helpful advice.

Construction of the algebra

The construction is an application of the method shown in [1]. We shall also keep the notation of that paper.

There was introduced a notion of a *compact domain* there as a system of N-tuples of bounded operators. More precisely the family D of sets $D(\mathscr{H}) \subseteq \mathscr{B}(\mathscr{H})^N$ (where $\mathscr{H}$ denotes any separable Hilbert space) is called a *compact domain* if for every separable Hilbert space the following three conditions are fulfilled:

(1) For every $a \in D(\mathscr{H})$ and every $*$-representation

$$\pi \colon C^*(a) \to \mathscr{B}(\mathscr{K})$$

we have $\pi(a) \in D(\mathscr{K})$;

(2) If for a given $a \in \mathscr{B}(\mathscr{H})^N$ there exists a family $\{\pi_t\}_{t\in T}$ of representations of $C^*(a)$ such that $\bigcap_{t\in T} \operatorname{Ker} \pi_t = \{0\}$ and $\pi_t(a) \in D(\mathscr{H}_t)$ for all $t \in T$ (where $\mathscr{H}_t$ denotes the carrier Hilbert space of π_t) then $a \in D(\mathscr{H})$;

(3) There exists a constant $M > 0$ independent of $\mathscr{H}$ such that

$$\sup_{a\in D(\mathscr{H})} \|a_k\| \leq M.$$

If $a \in \mathscr{B}(\mathscr{H})^N$ and $a = (a_1, \dots, a_N)$ then $C^*(a)$ denotes the unital C^*-algebra generated by $a_1, \dots, a_N$. Similarly if π is a representation of $C^*(a)$ then $\pi(a)$ denotes $(\pi(a_1), \dots, \pi(a_N))$.

Let $C(D)$ be the unital C^*-algebra of *continuous functions* on D. F is an element of $C(D)$ if for any Hilbert space $\mathscr{H}$ and any $(a_1, \dots, a_N) \in D(\mathscr{H})$, $F(a_1, \dots, a_N)$ lies in $C^*(a)$, and for any representation π of $C^*(a)$ we have

$$F(\pi(B_0), \dots, \pi(B_N)) = \pi(F(B_0), \dots, F(B_N)).$$

For any separable Hilbert space $\mathscr{H}$ we set $\mathbf{D}(\mathscr{H}) = \{(B_0, \dots, B_N) \in \mathscr{B}(\mathscr{H})^{N+1} : B_0, \dots, B_N$ satisfy conditions (1)–(5)$\}$.

By virtue of [1, Theorem 1.3] we get

Proposition. $\mathbf{D} = \{\mathbf{D}(\mathscr{H}) : \mathscr{H}$ *—separable Hilbert space*$\}$ *is a compact domain.*

We define $C_0(\mathbf{D})$ as those elements F belonging to $C(\mathbf{D})$ for which $F(0, \dots, 0) = 0$. $C_0(\mathbf{D})$ is clearly a C^*-subalgebra of $C(\mathbf{D})$ (without unity). We shall prove the following

Theorem. *There is a natural* **-isomorphism of the group* C^**-algebra* $C^*(\mathbf{G})$ *onto* $C_0(\mathbf{D})$.

Continuous functions on a compact space vanishing at a single point form a commutative C^*-algebra which topologically corresponds to a locally compact space. One can now view at the group C^*-algebra of a simply connected lie group as a noncommutative C^* algebra of *continuous functions* on a *locally compact domain* vanishing at a single "point".

We precede the proof of the theorem with two lemmas.

Lemma 1. *Let* $\mathscr{H}$ *be a Hilbert space and* $(B_0, \dots, B_N)$ *be operators in* $\mathscr{B}(\mathscr{H})$ *satisfying conditions* (1)–(5). *Assume that* $\ker B_0 = \{0\}$. *Then there exists a strongly continuous unitary representation* U *of the group* $\mathbf{G}$ *acting on* $\mathscr{H}$ *such that*

$$\overline{dU(X_m)} = \overline{B_m B_0^{-1}}, \qquad m = 1, \dots, N.$$

Proof. Setting $A_m = \overline{B_m B_0^{-1}}$ for $m = 1, \dots, N$ and $\mathscr{D} = \mathrm{Dom}(B_0^{-1})$ we shall check that the following conditions hold:

1. A_m are skew-symmetric (i.e. $A_m \subseteq -A_m^*$), $m = 1, \dots, N$;
2. $\mathscr{D} \subseteq \mathrm{Dom}(A_m A_n)$, $m, n = 1, \dots, N$;
3. for any $\xi \in \mathscr{D}$ we have $A_m A_n \xi - A_n A_m \xi = \sum_{k=1}^N c_{mn}^k A_k \xi$, $m, n = 1, \dots, N$;
4. $\sum_{k=1}^N A_k^2$ is essentially selfadjoint on $\mathscr{D}$.

1. If ξ, $\varphi \in \mathrm{Dom}(B_0^{-1})$ then there exist $\tilde{\xi}$, $\tilde{\varphi}$ such that $B_0\tilde{\xi} = \xi$, $B_0\tilde{\varphi} = \varphi$, hence;

$$\begin{aligned}(A_m\xi, \varphi) &= (B_m B_0^{-1}\xi, \varphi) = (B_m B_0^{-1} B_0\tilde{\xi}, B_0\tilde{\varphi}) = (B_0\tilde{\xi}, B_0\tilde{\varphi}) \\ &= (\tilde{\xi}, B_m^* B_0\tilde{\varphi}) \qquad \text{(from (2)—the second condition on } B_m) \\ &= -(\tilde{\xi}, B_0 B_m\tilde{\varphi}) \\ &= -(B_0\tilde{\xi}, B_m B_0^{-1} B_0\tilde{\varphi}) = -(\xi, B_m B_0^{-1}\varphi) = -(\xi, A_m\varphi).\end{aligned}$$

2. We must show that $B_n(\mathscr{H}) \subseteq \mathrm{Dom}(\overline{B_m B_0^{-1}})$. For any $k = 1, \dots, N$, $B_k B_0^{-1/2}$ is bounded. Indeed

$$\begin{aligned}\|(B_k B_0^{-1/2})B_0\xi\|^2 &= (B_k^* B_k B_0^{1/2}\xi, B_0^{1/2}\xi) \qquad \text{(from (3))} \\ &\le (B_0 B_0^{1/2}\xi, B_0^{1/2}\xi) = \|B_0\xi\|^2.\end{aligned}$$

Therefore $B_0^{-1/2}B_k^*$ is also bounded, so $B_k^*(\mathscr{H}) \subseteq \mathrm{Dom}(B_0^{-1/2})$ and (5) now implies that $B_n(\mathscr{H}) \subseteq \mathrm{Dom}(B_0^{-1/2})$. Hence $\overline{B_m B_0^{-1}} B_n = (\overline{B_m B_0^{-1/2}})(B_0^{-1/2} B_n)$ is everywhere defined (as a product of two bounded operators).

3. It is enough to show that for any ξ, $\varphi \in \mathscr{H}$

$$((A_m A_n - A_n A_m)B_0\xi, B_0\varphi) = \left(\sum_{k=1}^{N} c_{mn}^k A_k B_0\xi, B_0\varphi\right).$$

We know that A_m, A_n are skew-symmetric and $B_0\varphi$ is in their domains, so

$$\begin{aligned}((A_m A_n - A_n A_m)B_0\xi, B_0\varphi) &= -(A_n B_0\xi, A_m B_0\varphi) + (A_m B_0\xi, A_n B_0\varphi) \\ &= -(\overline{B_n B_0^{-1}} B_0\xi, \overline{B_m B_0^{-1}} B_0\varphi) + (\overline{B_m B_0^{-1}} B_0\xi, \overline{B_n B_0^{-1}} B_0\varphi) \\ &= ((-B_m^* B_n + B_n^* B_m)\xi, \varphi).\end{aligned}$$

On the other hand

$$\left(\sum_{k=1}^{N} c_{mn}^k A_k B_0\xi, B_0\varphi\right) = \left(\sum_{k=1}^{N} c_{mn}^k \overline{B_k B_0^{-1}} B_0\xi, B_0\varphi\right) = \left(\sum_{k=1}^{N} c_{mn}^k B_0 B_n\xi, \varphi\right)$$

and the equality holds (from (4)).

4. $\sum_{k=1}^{N} A_k^2$ is essentially selfadjoint on $\mathscr{D}$ because it equals

$$I - B_0^{-1}|_{\mathscr{D}}.$$

Now the proof follows from Nelson's integrability theorem [2. Corollary 9.1]. □

We are now able to prove the following basic lemma.

Lemma 2. *There exist elements* $b_0, \dots, b_N$ *of* $C^*(\mathbf{G})$ *such that*:

1. *For any operators* $B_0, \dots, B_N$ *(on a Hilbert space) satisfying conditions* (1)–(5) *there is a* $*$*-homomorphism* U *of* $C^*(\mathbf{G})$ *onto the* C^**-algebra generated by* $B_0, \dots, B_N$*, such that*

$$U(b_k) = B_k, \qquad k = 0, 1, \dots, N.$$

2. $b_0, \dots, b_N$ *generate* $C^*(\mathbf{G})$.

Proof. Considering the left regular representation of $\mathbf{G}$ acting on $L^2(\mathbf{G})$ we know (cf. [3, Theorem 3.1]) that for any $X \in \mathfrak{g}$ the operator $X + I - \sum_{k=1}^N X_k^2$ (acting on smooth functions with compact support) is invertible and the inverse operator has a kernel in $L^1(\mathbf{G})$. We define b_0 and p_m $(m = 1, \dots, N)$ as the images of the kernels of $\overline{(I - \sum_{k=1}^N X_k^2)}^{-1}$ and $\overline{(X_m + I - \sum_{k=1}^N X_k^2)}^{-1}$ respectively under the canonical inclusion of $L^1(\mathbf{G})$ into $C^*(\mathbf{G})$. For each natural number n let ψ_n be the following function:

$$\psi_n(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq \frac{1}{n}, \\ n^2 & \text{if } x < \frac{1}{n}, \end{cases} \qquad x \in \mathbf{R}.$$

Of course $\psi_n(|p_m|)$ is a multiplier on $C^*(\mathbf{G})$. We shall show that $\psi_n(|p_m|)p_m^* \times (b_0 - p_m)$ is a Cauchy sequence in $C^*(\mathbf{G})$ (its limit will be denoted by b_m). To do that it will be enough to prove that for any nondegenerate representation U of $C^*(\mathbf{G})$ the sequence

$$(U(\psi_n(|p_m|)p_m^*(b_0 - p_m)))^* = (U(b_0) - U(p_m)^*)U(p_m)\psi_n(|U(p_m)|)$$

is norm convergent.

Because we can extend U to a representation of the group $\mathbf{G}$, we may also consider operators $A_1, \dots, A_N$, $B_0, \dots, B_N$ constructed for U as in the introduction. It is easy to establish that $U(b_0) = B_0$ and $U(p_m) = \overline{(A_m + B_0^{-1})}^{-1}$. We denote $U(p_m)$ by P_m.

By virtue of [2, Theorem 6.3] there exists a positive constant M such that $\|P_m^{*-1}\xi\| \leq M^2\|B_0^{-1}\xi\|$ for any $\xi \in \mathscr{E}_\infty$ (where $\mathscr{E}_\infty$ are the smooth vectors of U). So $\|B_0\xi\| \leq M^2\|P_m\xi\| = M^2\||P_m|\xi\|$ and operator monotony of the square root function implies that $\|B_0^{1/2}\xi\| \leq M\||P_m|^{1/2}\xi\|$ for any $\xi \in \mathscr{H}$ ($\mathscr{H}$—the carrier Hilbert space of U). In the proof of the previous lemma we showed that $B_m^* B_0^{-1/2}$ is bounded, so there is $\widetilde{C} > 0$ such that for all ξ in $\mathscr{H}$ we have $\|B_m^*\xi\| \leq \widetilde{C}\|B_0^{1/2}\xi\|$. Therefore $\|B_m^*\xi\| \leq \widetilde{C}\|B_0^{1/2}\xi\| \leq M\widetilde{C}\||P_m|^{1/2}\xi\|$. We write C instead of $M\widetilde{C}$.

If we take ξ from $P_m^{*-1}(\mathscr{E}_\infty)$ then

$$(P_m^* - B_0)\xi = B_0(B_0^{-1} - P_m^{*-1})P_m^*\xi = B_0 A_m P_m^*\xi = -B_m^* P_m^*\xi,$$

hence

$$
\begin{aligned}
B_m^* P_m^* &= B_0 - P_m^*, \\
(U(b_0) - U(p_m)^*)U(p_m)\psi_n(|U(p_m)|) &= (B_0 - P_m^*)P_m\psi_n(|P_m|) \\
&= B_m^*|P_m|^2\psi_n(|P_m|) = B_m^*\varphi_n(|P_m|),
\end{aligned}
$$

where φ_n is a real function such that $\varphi_n(x) = 1$ if $x > \frac{1}{n}$ and $\varphi_n(x) = n^2x^2$ if $x \le \frac{1}{n}$. For any $\xi \in \mathscr{H}$ we have:

$$
\begin{aligned}
\|B_m^*\varphi_n(|P_m|)\xi - B_m^*\xi\| &= \|B_m^*(\varphi_n - 1)(|P_m|)\xi\| \\
&\le C\||P_m|^{1/2}(\varphi_n - 1)(|P_m|)\xi\| \le Cn^{-1/2}\|\xi\|.
\end{aligned}
$$

Therefore $B_m^*\varphi_n(|P_m|) \xrightarrow{\|\ \|} B_m^*$, so the elements $b_1, \dots, b_N$ also exist.

Given $B_0, \dots, B_N$—operators on a Hilbert space satisfying conditions (1)–(5), we see (taking into account (3)) that on $\ker B_0$ all of them are equal to zero and the same holds for the adjoints (from (5)). The operators restricted to the orthogonal complement of $\ker B_0$ fulfill all the assumptions of Lemma 1. By virtue of that lemma we get a representation U of $\mathbf{G}$ such that $\overline{dU(X_m)} = \overline{B_m B_0^{-1}}$. It follows from the above considerations that U extended to $C^*(\mathbf{G})$ must carry b_m into B_m.

It remains to show that $b_0, \dots, b_N$ generate $C^*(\mathbf{G})$. If that is not so, we get two different representations π_1, π_2 of $C^*(\mathbf{G})$ (acting on the same Hilbert space) which are identical on $b_0, \dots, b_N$. From Lemma 1 we know that $\pi_1(b_0), \dots, \pi_1(b_N)$ gives rise to a representation of the group. That is why π_1 and π_2 as the corresponding representations of the algebra must coincide. This completes the proof. Q.E.D.

Now we are able to complete the proof of the main Theorem.

Let $B^m\colon (B_0, \dots, B_N) \mapsto B_m$ be the *coordinate functions* on $\mathbf{D}$. By virtue of [1, Theorem 3.4] $C_0(\mathbf{D})$ is generated by $B^0, \dots, B^N$. It is an immediate consequence of Lemma 2 that the mapping $b_m \mapsto B^m$ can be extended to a $*$-homomorphism Γ of $C^*(\mathbf{G})$ onto $C_0(\mathbf{D})$ (to see that one must apply part 1 of the lemma to the images of $B^0, \dots, B^N$ under a faithful representation of $C_0(\mathbf{D})$).

Clearly Γ has a trivial kernel. Indeed, if $\Gamma(c) = 0$ then for any cyclic representation π of $C^*(\mathbf{G})$ (acting necessarily on a separable Hilbert space $\mathscr{H}_\pi$) we have $\pi(c) = \Gamma(c)(\pi(b_0), \dots, \pi(b_N)) = 0$ (of course $(\pi(b_0), \dots, \pi(b_N)) \in \mathbf{D}(\mathscr{H}_\pi)$).

The proof is complete. Q.E.D.

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