

## ON PICARD'S THEOREM FOR ENTIRE QUASIREGULAR MAPPINGS

MATTI VUORINEN

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**ABSTRACT.** Several refinements of Picard's theorem for entire functions in the complex plane have been proved by many authors in connection with the theory of Picard sets. We prove a result of this type for entire quasiregular mappings in euclidean  $n$ -space in the case when the "Picard set" consists of a sequence  $a_k$  on a ray emanating from 0 with  $|a_k| = 2^k$ .

### 1. INTRODUCTION

Several results of geometric function theory have their multidimensional analogues for quasiregular mappings in  $R^n$  ([V3], [Ri2]). According to recent results of S. Rickman [Ri1], [Ri3], the Picard and Schottky theorems in particular have their multidimensional counterparts in this context.

Extending the classical Picard theorem, O. Lehto introduced in 1958 [L] the notion of a Picard set of an entire or meromorphic function of the complex plane. Some delicate properties of these sets were found by L. Carleson [C] and K. Matsumoto [M1], [M2] soon after the publication of [L]. Later contributions to the theory of Picard sets include works by many authors, most notably S. Toppila ([T1]–[T5]) and J. Winkler ([W1], [W2]) (for further information see the bibliography in [T4]).

The goal of this paper is to prove the following theorem, which can be formulated in terms of Picard sets in the special case of entire functions. (We shall not need the notion of a Picard set in this paper.) Let  $e_1 = (1, 0, \dots, 0) \in R^n$  and let  $p(n, K)$  be the integer in the multidimensional analogue of the Picard theorem ([Ri1]).

**1.1. Theorem.** *Let  $f: R^n \rightarrow R^n$  be a  $K$ -quasiregular mapping and  $W = \{a_1, \dots, a_p\}$ ,  $a_p = \infty$ , a set of distinct points in  $\overline{R}^n$  with  $p \geq p(n, K)$ . If  $f^{-1}W \subset \{2^k e_1: k = 1, 2, \dots\}$  then the limit  $\lim_{x \rightarrow \infty} f(x)$  exists.*

In the plane case, results stronger than Theorem 1.1 are known for entire functions. However, the previously known methods of studying Picard sets (such as Cauchy's integral formula) are not applicable to the present context.

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Instead our method relies on the uniform continuity properties of quasiregular mappings with respect to the quasihyperbolic metric. The key idea is contained in a lemma about removable isolated singularities of uniformly continuous functions, which yields a quantitative upper bound for the modulus of continuity of the extended mapping. This lemma is apparently new even for  $n = 2$ . Its proof relies essentially on the Harnack inequality, which in the present context follows from the results of J. Serrin [S] (cf. Yu. G. Reshetnyak [R]) and also uses S. Rickman's result in [Ri3].

2. PRELIMINARY RESULTS

We shall adopt the relatively standard notation and terminology of [V1]. The coordinate unit vectors in  $R^n$  are  $e_1, \dots, e_n$ . For  $x, y \in R^n$ , we denote  $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$  and similarly for half-open or open segments. For  $x \in R^n \setminus \{0\}$  let  $[x, \infty) = \{tx : t \geq 1\}$ . For  $x \in R^n$  and  $r > 0$  let  $B^n(x, r) = \{z \in R^n : |x - z| < r\}$ ,  $S^{n-1}(x, r) = \partial B^n(x, r)$ .  $B^n(r) = B^n(0, r)$ ,  $S^{n-1}(r) = \partial B^n(r)$ ,  $B^n = B^n(1)$ , and  $S^{n-1} = \partial B^n$ .

For the definition and the basic properties of the modulus  $M(\Gamma)$  of a curve family  $\Gamma$  in  $\bar{R}^n = R^n \cup \{\infty\}$ , the reader is referred to [V1]. The definitions of quasiconformal ( $qc$ ),  $K$ -quasiconformal ( $K$ - $qc$ ), quasiregular ( $qr$ ),  $K$ -quasiregular, quasimeromorphic ( $qm$ ), and  $K$ -quasimeromorphic ( $K$ - $qm$ ) mappings can be found in [MRV1], [MRV2], [Vu3].

We shall require several metrics. The spherical metric  $\sigma$  on  $\bar{R}^n$  is defined by the element of length

$$(2.1) \quad d\sigma = \frac{|dx|}{1 + |x|^2}.$$

The spherical chordal metric  $q$  is defined by

$$(2.2) \quad \begin{aligned} q(x, y) &= |x - y|((1 + |x|^2)(1 + |y|^2))^{-\frac{1}{2}}; & x \neq \infty \neq y \\ q(x, \infty) &= (1 + |x|^2)^{-\frac{1}{2}}. \end{aligned}$$

These two metric are equivalent, in fact  $q(x, y) = \sin \sigma(x, y)$ ,  $1 \leq \sigma(x, y) / q(x, y) \leq \pi/2$  for distinct  $x, y \in \bar{R}^n$  and  $q(0, \infty) = 1 = 2\sigma(0, \infty) / \pi$ . Both  $\sigma$  and  $q$  are invariant under a subgroup of the group  $GM(\bar{R}^n)$  of Möbius transformations, called spherical isometries. If  $X \subset \bar{R}^n$ , then  $GM(X) = \{f \in GM(\bar{R}^n) : fX = X\}$ .

The hyperbolic (or Poincaré) metric  $\rho$  in  $B^n$  is defined by the formula

$$(2.3) \quad \tanh^2(\rho(x, y)/2) = |x - y|^2 / (|x - y|^2 + (1 - |x|^2)(1 - |y|^2))$$

for all  $x, y \in B^n$ . A basic fact is that  $\rho$  is  $GM(B^n)$ -invariant.

Throughout the paper  $p$  will be a positive integer with  $p \geq 2$ . In most cases  $p \geq p(n, K)$ , where  $p(n, K)$  is the integer in the Picard and Schottky theorem for  $K$ - $qm$  mappings in  $R^n$ . Next we shall give a construction of a metric  $\tau$  following closely S. Rickman's exposition [Ri3]. A difference is that we

introduce a constant  $A$  in formula (2.6) below to ensure a monotone property of  $\tau$  which will be convenient later on. Therefore our metric  $\tau$  differs slightly from S. Rickman's metric  $\tau$ . It is easy to show, however, that these two metrics are equivalent.

For distinct points  $a_1, \dots, a_p$  in  $\overline{R}^n$  let  $Y = \overline{R}^n \setminus \{a_1, \dots, a_p\}$ ,  $\partial Y = \{a_1, \dots, a_p\}$ , and let

$$(2.4) \quad \beta = \frac{1}{4} \min\{\sigma(a_i, a_j) : 1 \leq i, j \leq p, i \neq j\}.$$

It follows from (2.4) that

$$(2.5) \quad \sigma(x) = \sigma(x, \partial Y) \leq \frac{\pi}{2} - 2\beta$$

for all  $x \in Y$ . Denote  $B_\sigma(z, r) = \{w \in \overline{R}^n : \sigma(z, w) < r\}$  for  $z \in \overline{R}^n$  and  $r \in (0, \pi/2]$ . Let  $U_j = B_\sigma(a_j, \beta) \setminus \{a_j\}$ ,  $j = 1, \dots, p$ .  $U = \bigcup_{j=1}^p U_j$ . A metric  $\tau_Y = \tau$  on  $Y$  is defined by the element of length

$$(2.6) \quad \begin{aligned} d\tau(x) &= \frac{d\sigma}{\sigma(x, a_j) \log(A/\sigma(x, a_j))}; & x \in U_j, & \quad j = 1, \dots, p \\ d\tau(x) &= \frac{d\sigma}{\beta \log(A/\beta)}; & x \in \overline{R}^n \setminus U. \end{aligned}$$

The constant  $A$  is chosen so that  $d\tau(x)/d\sigma$  is monotonically decreasing as a function of  $\sigma(x) = \sigma(x, \partial Y)$ . Since  $\sigma(x, a_j) \leq \pi$  we may choose  $A = \pi e$  (this choice is fixed throughout the paper). Explicitly,  $\tau(x, y)$  is defined by the formula

$$(2.6)' \quad \tau(x, y) = \inf_\gamma \int_\gamma d\tau(z)$$

where the infimum is taken over all curves  $\gamma$  with  $\sigma(\gamma) < \infty$  and  $x, y \in \gamma$ . It follows from the above monotone property of  $d\tau(x)$  that for all  $x, y \in Y$

$$(2.7) \quad Q_1 \sigma(x, y) \leq \tau(x, y)$$

where  $Q_1 = 1/(\beta \log(A/\beta))$ . By integrating along circular arcs not intersecting  $U$ , we see that for all  $x, y \in \overline{R}^n \setminus U$

$$(2.8) \quad \tau(x, y) \leq Q_2 \sigma(x, y)$$

where  $Q_2 = \pi Q_1/2$ . Similarly, one can show that for  $r \in (0, \beta)$  and  $j = 1, \dots, p$

$$\tau(\partial B_\sigma(a_j, r)) \leq \pi/\log(A/r).$$

It follows from (2.6) that for  $1 \leq j \leq p$  and all  $x, y \in \overline{U}_j \setminus \{a_j\}$

$$(2.9) \quad \tau(x, y) \geq \left| \log \frac{\log(A/\sigma(x, a_j))}{\log(A/\sigma(y, a_j))} \right|.$$

Let  $t_z$  be a sense-preserving spherical isometry with  $t_z(z) = 0$ ,  $z \in \overline{R}^n$ . For convenient reference, we point out here that for  $\alpha \in (0, \pi/2)$ ,  $x, y \in \overline{R}^n$

$$(2.10) \quad \sigma(x, y) = \alpha \Leftrightarrow |t_x(y)| = \tan \alpha.$$

By (2.10) we get

$$(2.11) \quad B_\sigma(0, \alpha) = B^n(\tan \alpha).$$

It is also clear that for  $\sigma(x, y) = \pi/2$  and  $0 < \alpha < \pi/2$

$$(2.12) \quad B_\sigma(x, \alpha) = \overline{R}^n \setminus \overline{B}_\sigma(y, \pi/2 - \alpha).$$

In what follows,  $n \geq 2$ ,  $K \geq 1$ ,  $p \geq 2$ ,  $Y, \beta, A = \pi e$  will be fixed as defined above. Note that  $\tau_{hY}(h(x), h(y)) = \tau_Y(x, y)$  for spherical isometries  $h$ .

2.13. **Lemma.** *Let  $x, y \in U_j$ ,  $j = 1, \dots, p$ . Then*

$$\left| \tau(x, y) - \left| \log \frac{\log(A/\sigma(y, a_j))}{\log(A/\sigma(x, a_j))} \right| \right| \leq \pi Q_1$$

where  $Q_1$  is as in (2.7).

*Proof.* As pointed out in [Ri3, p. 138], a result of this form follows easily from the definitions.

2.14. **Lemma.** *For all  $x, y \in Y$  the following inequality holds*

$$\tau(x, y) \geq Q_3 \left| \log \frac{\log(A/\sigma(x))}{\log(A/\sigma(y))} \right|$$

where  $\sigma(z) = \sigma(z, \partial Y)$  and  $Q_3$  depends only on  $\beta$ .

*Proof.* By relabeling the points if necessary, we may assume  $\sigma(x) > \sigma(y)$ . Consider two cases.

Case A.  $\sigma(y) > \beta/2$ .

Let  $D = 1/\log(A/\sigma(x))$  and  $d = D \log(A/\sigma(y))$ . Then  $D < 1$  by (2.5). We obtain

$$d < D(\log(A/\sigma(x)) + \frac{A/\sigma(y) - A/\sigma(x)}{A/\sigma(x)}) = 1 + D \left( \frac{\sigma(x)}{\sigma(y)} - 1 \right)$$

and hence

$$(2.15) \quad \log d < D \left( \frac{\sigma(x)}{\sigma(y)} - 1 \right) \leq D \sigma(x, y) / \sigma(y).$$

In Case A, the inequality  $\sigma(y) > \beta/2$  holds, and therefore by (2.15) and by the proof of (2.7) we obtain

$$\log d \leq 2\tau(x, y) / (\beta \overline{Q}_1)$$

where  $\overline{Q}_1 = 2/(\beta \log(2A/\beta))$ . The desired inequality with  $Q_{3A} = 1/\log(2A/\beta)$  therefore holds in Case A.

Case B.  $\sigma(y) \leq \beta/2$ .

Let  $j$  be such that  $y \in U_j$ . If also  $x \in U_j$ , then the proof of Case B with  $Q'_{3B} = 1$  follows from (2.9). Otherwise  $x \in Y \setminus U_j$ . Let  $x_1 \in \partial U_j$  with  $\tau(x, y) \geq \tau(x_1, y)$ . Then by (2.9)

$$\tau(x, y) \geq \tau(x_1, y) \geq \log \frac{\log(A/\sigma(y))}{\log(A/\beta)}.$$

By (2.5) we get because  $A = \pi e$  and  $\sigma(x) < \pi$

$$\frac{\log(A/\sigma(y))}{\log(A/\sigma(x))} < \log(A/\sigma(y)).$$

The last two inequalities show that it suffices to find a constant  $E \in (0, 1]$  such that

$$\log \frac{\log(A/\sigma(y))}{\log(A/\beta)} \geq E \log(\log(A/\sigma(y))).$$

It is easy to show that we can choose  $E = (\log w_0)/\log(Cw_0) < 1$  where  $C = \log(A/\beta) > 2$  and  $w_0 = (\log(2A/\beta))/\log(A/\beta) > 1$ . Hence in Case B we may choose  $Q_{3B} = E$ .

In conclusion, in both Cases A and B we can choose  $Q_3 = \min\{Q_{3A}, Q_{3B}\}$ .

Lemma 2.14 is a modification of (2.9). Still another modification is needed.

2.16. **Lemma.** For all  $x, y \in Y$  and all  $z \in \partial Y$  the following inequality holds:

$$\tau(x, y) \geq Q_4 \left| \log \frac{\log(A/\sigma(x, z))}{\log(A/\sigma(y, z))} \right|$$

where  $Q_4$  depends only on  $\beta$ .

*Proof.* The proof is similar to the proof of Lemma 2.14 and the details are omitted.

2.17. **Quasihyperbolic metric.** Let  $G$  be a proper subdomain of  $R^n$ . The quasihyperbolic distance  $k_G(a, b)$  of  $a, b \in G$  is defined by [GP]

$$(2.18) \quad k_G(a, b) = \inf_{\gamma \in \Gamma_{ab}} \int_{\gamma} \frac{ds}{d(x, \partial G)}$$

where  $\Gamma_{ab}$  is the collection of all rectifiable curves  $\gamma$  in  $G$  with  $a, b \in \gamma$ . It is easy to see that  $k_G$  is a metric on  $G$ . For  $a, b \in G$  set

$$(2.19) \quad j_G(a, b) = \log \left( 1 + \frac{|a - b|}{\min\{d(a), d(b)\}} \right)$$

where  $d(x) = d(x, \partial G)$ . It follows from (2.19) that

$$(2.20) \quad \left| \log \frac{d(a)}{d(b)} \right| \leq j_G(a, b) \leq \left| \log \frac{d(a)}{d(b)} \right| + \log \left( 1 + \frac{|a - b|}{d(a)} \right)$$

for all  $a, b \in G$ . It is well known that  $j_G$  is a metric on  $G$ . As shown in [GP, (2.2)] the very useful inequality

$$(2.21) \quad k_G(a, b) \geq j_G(a, b)$$

holds for all  $a, b \in G$ . In the opposite direction we have

$$(2.22) \quad k_G(a, b) \leq \log \left( 1 + \frac{|a - b|}{d(a) - |a - b|} \right)$$

for  $b \in B^n(a, d(a))$  [Vu2, (2.32)]. Bernoulli's inequality

$$(2.23) \quad \log(1 + as) \leq a \log(1 + s)$$

for  $s > 0, a \geq 1$ , will also be useful. Now (2.22) and (2.23) together yield

$$(2.24) \quad k_G(a, b) \leq \frac{1}{1 - s} j_G(a, b)$$

for  $b \in \overline{B}^n(a, sd(a)), s \in (0, 1)$ . For some properties of  $k_G$  the reader is referred to [GP], [GO], [Vu3].

Let  $G \subset \mathbb{R}^n$  be a domain,  $G \neq \mathbb{R}^n$ . A mapping  $f: G \rightarrow \overline{\mathbb{R}^n}$  is said to be  $\omega$ -normal [Vu2] if

$$(2.25) \quad \sigma(f(x), f(y)) \leq \omega(k_G(x, y))$$

holds for all  $x, y \in G$  where  $\omega: [0, \infty) \rightarrow [0, \infty)$  is a homeomorphism with  $\omega(0) = 0$ . We call  $f$  normal if it is  $\omega$ -normal for some  $\omega$ . Sometimes the function  $\omega$  is denoted by  $\omega_f$ .

For  $n = 2, G = B^2$ , and  $f$  meromorphic, it is well known (and easy to prove) that this definition is equivalent to the definition of a normal meromorphic function given by Lehto and Virtanen [LV].

The next result shows that normality is a local property of the function.

**2.26. Lemma.** *Let  $G \subset \mathbb{R}^n$  be a domain and  $f: G \rightarrow \overline{\mathbb{R}^n}$  be continuous. Then  $f$  is normal if and only if there exists a homeomorphism  $\bar{\omega}: [0, \infty) \rightarrow [0, \infty)$  with  $\bar{\omega}(0) = 0$  such that*

$$\sigma(f(x), f(y)) \leq \bar{\omega}(k_B(x, y))$$

for every ball  $B = B^n(x, r)$  in  $G$  and for all  $y \in B$ .

*Proof.* The proof is a somewhat lengthy (although straightforward) discussion where (2.20)–(2.24) are useful. It should be noted that if  $\omega$  is given, then  $\bar{\omega}$  has a majorant depending only on  $\omega$  (thus the majorant is independent of  $n$  and of the geometric and topological properties of  $G$ ). The details are omitted.

Let  $\{m_D: D \subset \overline{\mathbb{R}^n}\}$  be a family of metrics. We say that this family is *monotone* if  $D_1 \subset D_2$  implies  $m_{D_1}(x, y) \geq m_{D_2}(x, y)$  for all  $x, y \in D_1$ . For instance,  $\{k_D: D \subset \mathbb{R}^n, D \neq \mathbb{R}^n \text{ is a domain}\}$  is a monotone family, and the same is true if  $k_D$  is replaced by  $j_D$  or  $\sigma_D$ , the restriction of  $\sigma$  to  $D$ .

**2.27. Lemma.** *Let  $G_1, G_2$  be domains in  $\mathbb{R}^n$  with  $G_1 \cap G_2 \neq \emptyset, G_1 \neq \mathbb{R}^n \neq G_2$  and assume that there exists  $c \in (0, 1)$  such that*

$$(1) \quad d(x, \partial G_1) + d(x, \partial G_2) \geq cd(x, \partial(G_1 \cup G_2)), \text{ for all } x \in G = G_1 \cup G_2.$$

Suppose that  $f: G \rightarrow fG$  is continuous,  $fG \subset \mathbb{R}^n$ ; that  $\{m_D: D \subset \mathbb{R}^n\}$  is a monotone family of metrics; and that

$$m_{fG_j}(f(x), f(y)) \leq \omega_j(k_{G_j}(x, y))$$

for  $x, y \in G_j$  and  $j = 1, 2$ . Then there exists  $\omega: [0, \infty) \rightarrow [0, \infty)$  such that

$$(2) \quad m_{fG}(f(x), f(y)) \leq \omega(k_G(x, y))$$

and  $\omega(t) \rightarrow 0$  as  $t \rightarrow 0$  provided  $\omega_j(t) \rightarrow 0$ , as  $t \rightarrow 0$ ,  $j = 1, 2$ .

*Proof.* We consider two cases. Let  $d(x) = d(x, \partial G)$ .

Case A.  $k_G(x, y) \leq \log(1 + c/4)$ .

In this case  $|x - y| \leq \frac{c}{4} \min\{d(x), d(y)\}$  by virtue of (2.21). We may assume  $d(x) \leq d(y)$ . By the hypothesis (1) of the lemma there exists  $i \in \{1, 2\}$  such that  $d(x, \partial G_i) \geq \frac{c}{2}d(x)$ , i.e.,  $B^n(x, cd(x)/2) \subset G_i$ . Fix such  $i$ . By relabeling (if necessary), we may assume that  $i = 1$ . Then

$$y \in B^n(x, c \min\{d(x), d(y)\}/4) \subset B^n(x, \frac{1}{2}d(x, \partial G_1)).$$

It follows from (2.24) that

$$k_{G_1}(x, y) \leq 2j_{G_1}(x, y) = 2 \log \left( 1 + \frac{|x - y|}{\min\{d_1(x), d_1(y)\}} \right)$$

where  $d_1(z) = d(z, \partial G_1)$ . By the above calculation,  $|x - y| \leq \frac{1}{2}d_1(x)$  and hence  $d_1(y) \geq \frac{1}{2}d_1(x)$ . The last inequality, together with (2.23) and (2.21), now yields

$$\begin{aligned} k_{G_1}(x, y) &\leq 2 \log \left( 1 + \frac{2|x - y|}{d_1(x)} \right) \\ &\leq 2 \log \left( 1 + \frac{4|x - y|}{cd(x)} \right) < \frac{8}{c} j_G(x, y) \\ &\leq \frac{8}{c} k_G(x, y). \end{aligned}$$

Conclusion: In Case A we have

$$(2.28) \quad \begin{aligned} m_{fG}(f(x), f(y)) &\leq m_{fG_1}(f(x), f(y)) \leq \omega_1(k_{G_1}(x, y)) \\ &\leq \omega_1 \left( \frac{8}{c} k_G(x, y) \right) \end{aligned}$$

where also the monotone property of the family  $\{m_D\}$  was applied.

Case B.  $k_G(x, y) > \log(1 + c/4)$ .

Fix a geodesic segment  $\gamma$  of the quasihyperbolic metric [GO] with  $x, y \in \gamma$  and points  $z_1, \dots, z_{p+1} \in \gamma$  with  $z_1 = x$ ,  $z_{p+1} = y$  and  $k_G(z_i, z_{i+1}) = \log(1 + c/4)$  for  $i = 1, \dots, p - 1$  and  $k_G(z_p, z_{p+1}) \leq \log(1 + c/4)$ , and with

$p \leq 1 + k_G(x, y)C$ ,  $C = 1/\log(1 + c/4)$ . Then by Case A

$$\begin{aligned}
 (2.29) \quad m_{fG}(f(x), f(y)) &\leq \sum_{j=1}^p m_{fG}(f(z_j), f(z_{j+1})) \\
 &\leq (1 + k_G(x, y)C) \max\{T_1, T_2\} \\
 &\quad \text{where } T_i = \omega_i\left(\frac{8}{c} \log(1 + c/4)\right).
 \end{aligned}$$

Hence in Case B we obtain by (2.29)

$$m_{fG}(f(x), f(y)) \leq 2C \max\{T_1, T_2\} k_G(x, y).$$

Finally, the desired function  $\omega$  is defined as follows with the aid of (2.28) and (2.29):

$$\begin{aligned}
 \omega(t) &= 2 \max \left\{ \omega_1 \left( \frac{8t}{c} \right), \omega_2 \left( \frac{8t}{c} \right) \right\}; \quad t \in (0, \log(1 + c/4)] \\
 \omega(t) &= (Ct + 1) \max\{T_1, T_2\}; \quad t > \log(1 + c/4).
 \end{aligned}$$

### 3. A REMOVABLE SINGULARITY LEMMA

We begin with the following result of S. Rickman [Ri3]:

**3.1. Lemma.** *For  $K \geq 1$  and each integer  $n \geq 3$  there exists a number  $\delta = \delta(n, K) > 0$  and a positive integer  $p_0 = p(n, K) \geq 3$  such that the following holds. If  $p \geq p_0$  and  $a_1, \dots, a_p \in \bar{R}^n$  are distinct points and if  $f: B^n \rightarrow Y$ ,  $Y = \bar{R}^n \setminus \{a_1, \dots, a_p\}$ , is  $K$ -q.m, then*

$$\tau(f(x), f(y)) \leq C_1 \max\{\rho(x, y), \delta\}$$

where  $\tau$  is the metric defined in (2.6) and  $C_1$  is a constant depending only on  $n, K$ , and  $\beta$  (cf. (2.4)).

Throughout this section we assume that  $p$  is as in Lemma 3.1. It is easy to show that  $\rho(x, y) \leq 2k_{B^n}(x, y) \leq 2\rho(x, y)$  for all  $x, y \in B^n$ . Therefore, in the definition of a normal function of the unit ball  $B^n$  we may, and shall, replace the quasihyperbolic metric  $k_{B^n}$  by the  $GM(B^n)$ -invariant metric  $\rho$ .

**3.2. Corollary.** *Let  $f: B^n \rightarrow Y$  be as in Lemma 3.1. Then  $f$  is normal and furthermore*

$$\sigma(f(x), f(y)) \leq C_2 \rho(x, y)^\alpha; \quad \alpha = K^{1/(1-n)}$$

for all  $x, y \in B^n$  where  $C_2$  depends only on  $n, K$ , and  $\beta$ .

*Proof.* The proof follows from (2.7), 3.1, and [Vu2, 5.8].

**3.3. Corollary.** *Let  $f: B^n \rightarrow Y$ , be as in Lemma 3.1 and  $z \in \{a_1, \dots, a_p\}$ . Then  $v_f^z(x) = \log(A/\sigma(f(x), z))$  satisfies a Harnack condition*

$$(3.4) \quad v_f^z(x) \leq C_3 v_f^z(y)$$

for all  $x, y \in B^n$  with  $|x - y| \leq (1 - |x|)/2$ , where  $C_3$  depends only on  $n, K$ , and  $\beta$ .

*Proof.* It follows easily from (2.3) that  $\rho(x, y) < \log 6$  for all  $x, y \in B^n$  with  $|x - y| \leq (1 - |x|)/2$ . By 2.16 and 3.1

$$Q_4 \log(v_f^z(x)/v_f^z(y)) \leq \tau(f(x), f(y)) \leq C_1 \max\{\delta, \log 6\}$$

for all  $x, y \in B^n$  with  $|x - y| \leq (1 - |x|)/2$  and hence (3.4) follows with  $C_3 = \exp\{C_1 \max\{\delta, \log 6\}/Q_4\}$ .

We next consider the problem of extending a  $K$ -qm mapping to an isolated singularity. We are mainly interested in finding a quantitative estimate for the modulus of continuity of the extended mapping.

**3.4. Lemma.** *Let  $Y$  be as in Lemma 3.1 and let  $f: B^n \setminus \{0\} \rightarrow Y$  be  $K$ -qm. Then  $f$  has a limit at 0. Denote by  $f^*$  the extended mapping. Then*

$$\sigma(f^*(x), f^*(y)) \leq C_4 \rho(x, y)^\alpha; \quad \alpha = K^{1/(1-n)}$$

for all  $x, y \in B^n$ , where  $C_4$  depends only on  $n, K$ , and  $\beta$ .

*Proof.* The (finite or infinite) limit

$$\lim_{x \rightarrow 0} f(x) = f^*(0)$$

exists by [Ri1, 1.2]. Consider two cases. We may assume that  $f$  is nonconstant.

*Case A.*  $f^*(0) \in Y$ .

In this case the assertion follows from 3.2 with the constant  $C_2$ .

*Case B.*  $f^*(0) \in \partial Y$ .

Since the left side of the asserted inequality is invariant under spherical isometries, we may assume that  $f^*(0) = a_1$ . By 3.3, the function

$$v_i(x) = \log(A/\sigma(f(x), a_i))$$

satisfies a Harnack condition in  $B^n \setminus \{0\}$ . That is, if  $x \in B^n \setminus \{0\}$  and  $B^n(x, 2r) \subset B^n \setminus \{0\}$ , then

$$(3.5) \quad \max_{\overline{B}^n(x, r)} v_i(z) \leq C_4 \min_{\overline{B}^n(x, r)} v_i(z); \quad 1 \leq i \leq p$$

where  $C_4$  depends only on  $n, K, \beta$ . Next it follows from (3.5) that [Vu1, 3.3]

$$(3.6) \quad \max_{|z|=t} v_i(z) \leq C_5 \min_{|z|=t} v_i(z); \quad t \in (0, \frac{1}{2}]$$

for  $i = 1, \dots, p$ , where  $C_5 \geq C_4$  depends only on  $n, K, \beta$ .

For  $t \in (0, \frac{1}{2}]$  let  $R_t = B^n(\frac{1}{2}) \setminus \overline{B}^n(t^2)$ . We shall find a quantitative upper bound for the spherical diameter  $\sigma(fS^{n-1}(t))$ ,  $t \in (0, \frac{1}{2}]$ .

For each  $t \in (0, \frac{1}{2}]$  choose  $x_t \in S^{n-1}(t)$  and let  $I(t)$  be the set of all those indices  $i$  for which  $\sigma(f(x_t), a_i) \geq \beta$ . Then  $\text{card } I(t) \geq p - 1$  and

$v_i(x_i) \leq \log(A/\beta)$  for  $i \in I(t)$ . It follows from (3.6) that if  $t \in (0, \frac{1}{2}]$ ,  $i \in I(t)$ , and  $|y| = t$ , then

$$\sigma(f(y), a_i) \geq A(\beta/A)^{C_5} = C_6.$$

Because  $p \geq 3$  and  $\text{card } I(t) \geq p - 1$  for all  $0 < t \leq \frac{1}{2}$ , it follows that  $I(\frac{1}{2}) \cap I(t^2) \neq \emptyset$  and hence there is  $j_t \in I(\frac{1}{2}) \cap I(t^2)$  for each  $t \in (0, \frac{1}{2}]$ . Because  $a_{j_t} \in \partial Y$ , it follows from the maximum principle (or from the fact that  $f$  is open) that

$$f(R_t) \cap B_\sigma(a_{j_t}, C_6) = \emptyset.$$

Let  $\Gamma'_t$  be the set of all curves  $\gamma: [0, 1] \rightarrow \bar{R}^n$  with  $\gamma(0) \in fS^{n-1}(t)$  and  $\gamma(1) \in \partial B_\sigma(a_{j_t}, C_6)$ ,  $t \in (0, \frac{1}{2}]$ . Denote by  $\Gamma_t$  the set of all maximal liftings of the elements of  $\Gamma'_t$ , starting at  $S^{n-1}(t)$  (for terminology, see [MRV3, 3.11]). It follows from [MRV3, 3.12] that each  $\alpha \in \Gamma_t$  intersects either  $S^{n-1}(\frac{1}{2})$  or  $S^{n-1}(t^2)$ ,  $0 < t \leq \frac{1}{2}$ . Therefore by a standard inequality [V1, 7.5]

$$(3.7) \quad M(\Gamma_t) \leq 2\omega_{n-1}(\log 1/2t)^{1-n}$$

for  $t \in (0, \frac{1}{2})$ . By performing an auxiliary Möbius transformation and a spherical symmetrization one can show that (cf. [Vu3, 7.32, (7.24), (2.27), 1.17])

$$(3.8) \quad M(\Gamma'_t) \geq \omega_{n-1}(\log(\lambda_n C_7 / \sigma(fS^{n-1}(t))))^{1-n}$$

where  $\lambda_n \in [4, 2e\alpha^{n-1})$  is a constant depending only on  $n$  and  $C_7 = 4(1 - C_6^2)C_6^{-2}$ . Because  $M(\Gamma'_t) \leq KM(\Gamma_t)$  by [V2], it follows from (3.7) and (3.8) that

$$(3.9) \quad \sigma(fS^{n-1}(t)) \leq C_7 \lambda_n (2t)^\alpha; \quad \alpha = (2K)^{1/(1-n)}$$

for all  $t \in (0, \frac{1}{2})$ . Hence there exists a number  $t_0 \in (0, \frac{1}{2})$  depending only on  $n$ ,  $K$ , and  $\beta$  such that

$$f^*B^n(t) \subset B_\sigma(f^*(0), \sigma(fS^{n-1}(t))) \subset B_\sigma(a_1, \beta)$$

whenever  $t \in (0, t_0]$ .

It follows from 2.26 and 3.2 that  $f|B^n \setminus \{0\}$  is  $\omega_1$ -normal where  $\omega_1(t) = C_8 t^\alpha$  and  $C_8$  depends only on  $n, K, \beta$ . On the other hand,  $f^*|B^n(t_0)$  is  $\omega_2$ -normal with  $\omega_2(t) = C_9 t^\alpha$  by [Vu2, 5.8] where  $C_9$  depends only on  $n, K, \beta$ . Finally, by the proof of Lemma 2.27 we see that  $f^*$  is  $\omega$ -normal in  $B^n$  with  $\omega(t) = 2 \max\{\omega_1(8t/t_0), \omega_2(8t/t_0)\}$  for small  $t$ .

As a corollary we obtain a result due to L. Carleson [C] (see also [M2]) in the case of meromorphic functions.

**3.10. Corollary.** *Let  $K \geq 1$ ,  $Y$  as in Lemma 3.1, let  $\theta > 0$ ,  $R(1, e^{2\theta}) = \{z \in R^n : 1 < |z| < e^{2\theta}\}$ , and let  $f: R(1, e^{2\theta}) \rightarrow Y$  be  $K$ -qm. Then*

$$\sigma(fS^{n-1}(e^\theta)) \leq C_{10} e^{-\theta\alpha}; \quad \alpha = (2K)^{1/(1-n)},$$

where  $C_{10}$  depends only on  $n, K, \beta$ .

*Proof.* Apply the proof of Lemma 3.4 to a slightly smaller annulus.

## 4. THE PROOF OF THEOREM 1.1

In this section we prove the main result of this paper.

**4.1. Theorem.** *Let  $\omega: [0, \infty) \rightarrow [0, \infty)$  be a homeomorphism with  $\omega(0) = 0$ , let  $(b_i)$  be a sequence in  $\mathbb{R}^n$  with  $|b_i| = 2^i$ , and let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $K$ - $q$ -mapping such that  $|f(b_i)| < 1$  for all  $i = 1, 2, \dots$ , and the following holds. For each  $i = 1, 2, \dots$ , the mapping  $f|_{A_i}$  with  $A_i = B^n(2^{i+1}) \setminus \overline{B}^n(2^{i-1})$ , is  $\omega$ -normal. Then  $f$  is a constant.*

*Proof.* From the proof of [Vu2, 5.13] and [Vu1, 3.3] it follows that  $|f(x)| < D_1 < \infty$  for all  $x \in \bigcup_{i=1}^{\infty} S^{n-1}(2^i)$ . If  $f$  is a constant there is nothing to prove. Otherwise  $f$  is open and hence  $|f(x)| < D_1$  for all  $x \in \mathbb{R}^n$ . But this contradicts the Liouville theorem in [MRV2, 3.9].

**4.2. Proof of Theorem 1.1.** If  $f^{-1}W$  is bounded, the limit exists by [Ri1, 1.2]. Thus we may assume that  $f^{-1}W$  is not bounded, and without loss of generality we may assume that  $f^{-1}W = \{2^k e_1: k = 1, 2, \dots\}$ . Let

$$A_i = B^n(2^{i+1}) \setminus \overline{B}^n(2^{i-1}), \quad A_i^0 = A_i \setminus \{2^i e_1\}.$$

From 2.26 it follows that  $f|_{A_i^0}$  is  $\omega$ -normal where  $\omega$  depends only on  $n, K, \beta$ , and from 3.4 and 2.26 we also conclude that  $f|_{A_i}$  is  $\omega_1$ -normal where  $\omega_1$  depends only on  $n, K$ , and  $\beta$ . Thus after a rescaling the hypotheses of Theorem 4.1 are satisfied, and accordingly  $f$  is a constant and the limit exists.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, HELSINKI, FINLAND