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ON DUAL SPACES WITH BOUNDED SEQUENCES WITHOUT WEAK* CONVERGENT CONVEX BLOCKS

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ABSTRACT. In this work we show that if X^* contains bounded sequences without weak* convergent convex blocks, then it contains an isometric copy of $L_1(\{0,1\}^{\omega_1})$.

1. INTRODUCTION

We are concerned with the relation between properties of the weak* topology of the dual X^* of a Banach space X and the property of X containing $\ell_1(\Gamma)$, or of X^* containing $L_1(\{0,1\}^\Gamma)$ for a set Γ. The results of this manuscript are related to those of J. Bourgain [B], R. Haydon [Hy], R. Haydon, M. Levy and E. Odell [HLO] and J. Hagler and W. B. Johnson [HJ]; in particular, they generalize results obtained in [B, Hy, HJ].

The notations and terminology are mostly standard. The first infinite ordinal is denoted by ω_0; the first uncountable by ω_1 and the first ordinal with the cardinality of the continuum, by ω_c. The ordinal ω_p is taken to be the smallest ordinal such that there exists a family $(N_\xi)_{\xi<\omega_p}$ of infinite subsets of $\mathbf{N}$ having the property that $\bigcap_{\xi\in F} N_\xi$ is infinite for every finite $F \subset \omega_p$, but not admitting an infinite $N \subset \mathbf{N}$, such that $N \backslash N_\xi$ is finite for each $\xi < \omega_p$. It is easy to see that, $\omega_1 \leq \omega_p \leq \omega_c$. More about ω_p can be found in [F]; it is known for example, that $\omega_1 < \omega_p = \omega_c$ if we assume $\neg$CH and MA by their definition $\omega_0, \omega_1, \omega_p$, and ω_c are initial ordinals and can so be identified with cardinals. Only for technical reasons do we distinguish between the finite ordinals and the elements of the positive integers $\mathbf{N}$, which we consider as cardinals.

For a set Γ, the cardinality is denoted by $|\Gamma|$; and $\mathscr{P}_f(\Gamma)$ and $\mathscr{P}_\infty(\Gamma)$ denote the set of all finite and infinite subsets of Γ, whereas $\mathscr{P}(\Gamma)$ denotes the power set. For simplicity, we consider only Banach spaces over the real field $\mathbf{R}$; for a Banach space X, $B_1(X)$ shall mean the unit ball and X^*, the dual

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space of X. The weak topology on X and the weak*- topology on X^* are also denoted by $\sigma(X, X^*)$ and $\sigma(X^*, X)$ respectively.

For a set Γ, $L_1(\{0,1\}^\Gamma)$ is the L_1-space for the product measure

$$\bigotimes_{\gamma\in\Gamma} \frac{1}{2}(\delta_0 + \delta_1)$$

on the set $\{0,1\}^\Gamma$ furnished with the product σ-algebra $\bigotimes_{\gamma\in\Gamma} \mathscr{P}(\{0,1\})$. We consider the following two properties of a Banach space X concerning the weak* topology on X^*:

We say that the Banach space X satisfies

(CBH) (convex block hypothesis) if X^* contains a bounded sequence (x'_n) which has no $\sigma(X^*, X)$-convergent convex block, and

(ACBH) (absolutely convex block hypothesis) if X^* contains a bounded sequence (x'_n) which has no $\sigma(X^*, X)$-convergent absolutely convex block basis,

where a sequence of the form $(\sum_{i-k_n}^{k_{n+1}-1} a_i x'_i : n \in \mathbf{N})$ is called a convex block (respectively an absolutely convex block basis) of (x'_n) if (k_n) is increasing in $\mathbf{N}$, $(a_n) \subset \mathbf{R}_0^+$ (respectively $(a_n) \subset \mathbf{R}$), and $\sum_{i=k_n}^{k_{n+1}-1} a_i = 1$ (respectively $\sum_{i=k_n}^{k_{n+1}-1} |a_i| = 1$) for each $n \in \mathbf{N}$.

It is obvious that (ACBH) implies (CBH) and we remark that (ACBH) is equivalent to the condition, considered by J. Hagler and W. B. Johnson [HJ] and by R. Haydon [Hy], that X^* contains an infinite-dimensional subspace Y in which $\sigma(X^*, X)$-convergence of sequences implies norm convergence. In [HJ] it was first observed that nonreflexive Grothendieck spaces enjoy (ACBH) and it was proven that (ACBH) implies that X contains an isometric copy of ℓ_1. R. Haydon [Hy] improved this result by showing that (ACBH) implies that $L_1(\{0,1\}^{\omega_p})$ is isometrically embedded in X^*. J. Bourgain and J. Diestel showed in [BD] that spaces having limited sets [cf. §3] which are not relatively weakly compact have the property (CBH) and in [B] it was shown that (CBH) implies that X contains an isometric copy of ℓ_1. Finally it was proven in [HLO] that under the (set-theoretical) assumption that $\omega_1 < \omega_p$ (CBH) implies that X contains a copy of $\ell_1(\omega_p)$, which is under this hypothesis equivalent to $L_1(\{0,1\}^{\omega_p}) \subset X^*$ [ABZ]; the nonreflexive Grothendieck space constructed in [T] under CH does not contain any copy of $\ell_1(\omega_1)$ and, thus, shows that the result in [HLO] is dependent on further set-axioms.

Our main purpose is to show:

1. **Theorem.** *If X has property (CBH), then X^* contains an isometric copy of $L_1(\{0,1\}^{\omega_1})$.*

Together with the above-cited result of [HLO] we deduce:

2. **Corollary.** *If X satisfies property (CBH), then X^* contains an isometric copy of $L_1(\{0,1\}^{\omega_p})$.*

2. Proof of theorem 1

The following lemma is due to H. P. Rosenthal [R]:

3. **Lemma** (cited from [HLO, p. 4, Lemma 3A]). *Let X satisfy (CBH). Then there exists a bounded sequence (x_n') in X^* and $c \in \mathbf{R}$ such that for every convex block (y_n') of (x_n') and every $\eta < \frac{1}{2}$ there exists an $x \in B_1(X)$ such that*

$$\limsup_{n\to\infty}\langle y_n', x\rangle > c + \eta, \quad \liminf_{n\to\infty}\langle y_n', x\rangle < c - \eta,$$

and

$$\sup_{\tilde{x}\in B_1(X)}\left[\limsup_{n\to\infty}\langle x_n', \tilde{x}\rangle - \liminf_{n\to\infty}\langle x_n', \tilde{x}\rangle\right] = 1.$$

For the sequel, we assume that X has property (CBH) and that we have chosen $(x_n') \subset X^*$ and $c \in \mathbf{R}$ as in Lemma 3. To handle the space $L_1(\{0,1\}^\Gamma)$ for a nonempty set Γ, we need the following notations: For a set A, the set of all mappings $\varphi\colon A \to \{0,1\}$ will be denoted by 2^A; for $A' \subset A$ and $\varphi' \in 2^{A'}$, the set of all extensions of φ' onto the whole of A will be denoted by $2^{\varphi', A}$. The union $\bigcup\{2^A | A \in \mathscr{P}_f(\Gamma)\}$ is denoted by S_Γ and for the domain of $\varphi \in S_\Gamma$ we write $D(\varphi)$.

R. Haydon [Hy, p. 6, Lemma 3] provided the following characterization for a Banach space Y to contain an isometric copy of $L_1(\{0,1\}^\Gamma)$.

4. **Lemma.** *Let Y be a Banach space and Γ a set. Then Y contains an isometric copy of $L_1(\{0,1\}^\Gamma)$ if and only if there exists a family $(y_\varphi\colon \varphi \in S_\Gamma)$ in Y satisfying* (a) *and* (b) *as given below*:

(a) $$y_{\varphi'} = 2^{|A'|-|A|}\sum_{\varphi\in 2^{\varphi', A}} y_\varphi \qquad \text{for any } A \in \mathscr{P}_f(\Gamma),\, A' \subset A \text{ and } \varphi' \in 2^{A'}$$

(*since* $|2^{\varphi', A}| = 2^{|A|-|A'|}$, *this means that* $y_{\varphi'}$ *is the arithmetic mean of* $(y_\varphi\colon \varphi \in 2^{\varphi', A})$).

(b) $$\left\|\sum_{\varphi\in 2^A} a_\varphi y_\varphi\right\| = \sum_{\varphi\in 2^A}|a_\varphi| \qquad \text{for any } A \in \mathscr{P}_f(\Gamma) \text{ and } (a_\varphi\colon \varphi \in 2^A) \subset \mathbf{R}.$$

In this case, there is an isometry $T\colon L_1(\{0,1\}^\Gamma) \to Y$ such that $T(e_\varphi) = y_\varphi$ for $\varphi \in S_\Gamma$, where $e_\varphi \in L_1(\{0,1\}^\Gamma)$ is defined by

$$e_\varphi := 2^{|D(\varphi)|}\chi_{\{\theta\in 2^\Gamma | \theta(\gamma)=\varphi(\gamma) \text{ if } \gamma\in D(\varphi)\}}.$$

Another sufficient condition, for X^* to contain $L_1(\{0,1\}^\Gamma)$ can be formulated using the following definition.

Definition. Let Γ be a set. A family $F = (x(A,B)\colon A \in \mathscr{P}_f(\Gamma), B \subset 2^A)$ in $B_1(X)$ is said to satisfy $(\mathscr{F}_\Gamma)$ if the following condition holds:

$(\mathscr{F}_\Gamma)$ For every $A \in \mathscr{P}_f(\Gamma)$ and $n \in \mathbf{N}$ there exists a family $(x'(\varphi,n)\colon \varphi \in 2^A) \subset C^*$ such that

$$\text{(a)} \qquad x'(\varphi,n) \in \mathrm{co}(\{x'_m | m \geq n\}), \qquad \text{if } \varphi \in 2^A,$$

and

(b)
$$\left\langle 2^{|A'|-|A|} \sum_{\varphi \in 2^{\varphi',A}} x'(\varphi,n), x(A',B') \right\rangle - c \begin{cases} \geq \frac{1}{2}(1 = \frac{1}{|A'|+1} - \frac{1}{n}) & \text{if } \varphi' \in B', \\ \leq -\frac{1}{2}(\frac{1}{|A'|+1} - \frac{1}{n}) & \text{if } \varphi' \notin B', \end{cases}$$

whenever $A' \subset A$, $\varphi' \in 2^{A'}$ and $B' \subset 2^{A'}$. For the sake of brevity, we will denote the set $\{(A,B) | A \in \mathscr{P}_f(\Gamma),\ B \subset 2^A\}$ by I_Γ, the set of all families $F = (x(A,B)\colon (A,B) \in I_\Gamma)$ which satisfy $(\mathscr{F}_\Gamma)$ by $\mathscr{F}_\Gamma$; and the values $\frac{1}{2}(1 - 1/(|A|+1) - 1/n)$ and $\frac{1}{2}(1 - 1/(|A|+1))$ by $\Delta(A,n)$ and $\Delta(A)$ respectively for $A \in \mathscr{P}_f(\Gamma)$ and $n \in \mathbf{N}$.

With these definitions we are in a position to state the following result.

5. **Lemma.** *Let Γ be an infinite set. If $\mathscr{F}_\Gamma \neq \emptyset$, then there exists an isometric copy of $L_1(\{0,1\}^\Gamma)$ in X^*.*

Proof. Let $F = (x(A,B)\colon (A,B) \in I_\Gamma) \subset B_1(X)$ satisfy property $(\mathscr{F}_\Gamma)$. For each $\varphi \in S_\Gamma$ and each $n \in \mathbf{N}$ choose $x'(\varphi,n) \in B_1(X^*)$ as prescribed in $(\mathscr{F}_\Gamma)$ and define for each $\psi \in S_\Gamma$ and each $A \in \mathscr{P}_f(\Gamma)$

$$\text{(5.1)} \qquad y'(\psi,A) := 2^{|D(\psi)\cap A|-|A|} \sum_{\varphi \in 2^{(\psi|_{D(\psi)\cap A}),A}} x'(\varphi,|A|+1).$$

The net $(y'(\psi,A)\colon \psi \in S_\Gamma)_{A \in \mathscr{P}_f(\Gamma)}$ has an accumulation point $(y'(\psi)\colon \psi \in S_\Gamma)$ in the product $K := \prod_{\varphi \in S_\Gamma} \overline{\mathrm{co}(\{x'_n\colon n \in \mathbf{N}\})}^{\omega^*}$, endowed with the product of the weak* topology on $\overline{\mathrm{co}(\{x'_n\colon n \in \mathbf{N}\})}^{\omega^*}$ (the elements of $\mathscr{P}_f(\Gamma)$ are ordered by inclusion). From $(\mathscr{F}_\Gamma)$ and (5.1), it follows that $(y'(\psi)\colon \psi \in S_\Gamma)$ fulfills the following three properties (5.2), (5.3) and (5.4):

$$\text{(5.2)} \qquad y'(\psi) \in \bigcap_{n \in \mathbf{N}} \overline{\mathrm{co}(\{x'_m\colon m \geq n\})}^{\omega^*} \qquad \text{for each } \psi \in S_\Gamma,$$

(5.3)
$$y'(\psi') = 2^{|A'|-|A|} \sum_{\psi \in 2^{\psi',A}} y'(\psi), \qquad \text{for } A' \subset A \in \mathscr{P}_f(\Gamma) \text{ and } \psi' \in 2^{A'},$$

$$\text{(5.4)} \qquad \langle y'(\psi), x(A,B)\rangle - c \begin{cases} \geq \Delta(A) & \text{if } \psi \in B, \\ \leq -\Delta(A) & \text{if } \psi \notin B, \end{cases}$$

for $A \in \mathscr{P}_f(\Gamma)$, $\psi \in 2^A$ and $B \subset 2^A$

[Since $y'(\psi)$ is a w^*-accumulation-point of the net $(y'(\psi, \tilde{A})\colon \tilde{A} \in \mathscr{P}_f(\Gamma)$, with $D(\psi) \subset \tilde{A})$.]

We now choose a fixed $\gamma \in \Gamma$. Since Γ is infinite, it suffices to show that the family $(y'(\psi^1) - y'(\psi^0))\colon \psi \in S_{\Gamma\setminus\{\gamma\}})$, satisfies (a) and (b) of Lemma 4, where for $\theta \in \{0, 1\}$, and $\psi \in S_{\Gamma\setminus\{\gamma\}}$ $\psi^\theta \in 2^{D(\psi)\cup\{\gamma\}}$, is given by $\psi^\theta|_{D(\psi)} = \psi$ and $\psi^\theta(\gamma) = \theta$. Condition (a) follows from (5.3). In order to show (b), let $A \in \mathscr{P}_f(\Gamma\setminus\{\gamma\})$ and $(a_\varphi\colon \varphi \in 2^A) \subset \mathbf{R}$. From (5.2) and Lemma 3 it follows that for any $x \in B_1(X)$ and $\varphi \in 2^A$ we have $\langle x, y'(\varphi^1) - y'(\varphi^0)\rangle \le 1$, which implies that $\|\sum_{\varphi\in 2^A} a_\varphi(y'(\varphi^1) - y'(\varphi^0))\| \le \sum_{\varphi\in 2^A} |a_\varphi|$. To show "$\ge$" let $\varepsilon > 0$. Without loss of generality, assume $2\Delta(A) \ge 1 - \varepsilon$. Otherwise replace A by an $\tilde{A} \in \mathscr{P}_f(\Gamma\setminus\{\gamma\})$ with $A \subset \tilde{A}$ and $2\Delta(\tilde{A}) \ge 1 - \varepsilon$ and note that by (5.3) we have

$$\sum_{\tilde{\varphi}\in 2^{\tilde{A}}} 2^{|A|-|\tilde{A}|} a_{(\tilde{\varphi}|_A)}(y'(\tilde{\varphi}^1) - y'(\tilde{\varphi}^0)) = \sum_{\varphi\in 2^A} a_\varphi(y'(\varphi^1) - y'(\varphi^0)).$$

Now take $x := x(A\cup\{\gamma\}, \{\varphi^1|\varphi \in 2^A \text{ and } a_\varphi \ge 0\}\cup\{\varphi^0|\varphi \in 2^A \text{ and } a_\varphi < 0\})$. By (5.4) we have

$$\begin{aligned}\left\|\sum_{\varphi\in 2^A} a_\varphi(y'(\varphi^1) - y'(\varphi^0))\right\| &\ge \sum_{\varphi\in 2^A} a_\varphi\langle x, y'(\varphi^1) - y'(\varphi^0)\rangle \\ &\ge \sum_{\varphi\in 2^A} a_\varphi \operatorname{sign}(a_\varphi) 2\Delta(A) \ge (1-\varepsilon)\sum_{\varphi\in 2^A} |a_\varphi|.\end{aligned}$$

The assertion follows since $\varepsilon \ge 0$ was arbitrary. □

By Lemma 5, it is enough to show that $\mathscr{F}_{\omega_1} \ne \varnothing$. As we will see from Lemma 6, it is sufficient to show that for every $\alpha \in [1, \omega_0]$ each $F \in \mathscr{F}_{[1,\alpha[}$ can be extended to an $F_0 \in \mathscr{F}_{[0,\alpha[}$.

6. **Lemma.** *Suppose that for every $\alpha \in [1, \omega_0]$, each family $F = (x(A, B)\colon (A, B) \in I_{[1,\alpha[}) \subset B_1(X)$ satisfying $(\mathscr{F}_{[1,\alpha[})$ can be extended to an $F_0 = (x(A, B)\colon (A, B) \in I_{[0,\alpha[})$ which satisfies $(\mathscr{F}_{[0,\alpha[})$. Then $\mathscr{F}_{\omega_1}$ is not empty; in particular, $L_1(\{0, 1\}^{\omega_1})$ can be embedded in X^*.*

Proof. In order to show that there exists an $F \in \mathscr{F}_{\omega_1}$, we define an $F_\beta \in \mathscr{F}_\beta$ by transfinite induction for every $\beta \in [0, \omega_1]$ such that $F_\beta|_{I_{\tilde{\beta}}} = F_{\tilde{\beta}}$ whenever $\tilde{\beta} < \beta$. If $\beta = \tilde{\beta} + 1$, with $\tilde{\beta} < \omega_1$ and with $F_{\tilde{\beta}} \in \mathscr{F}_{\tilde{\beta}}$ having been chosen, one can use the assumption to get an extension F_β of $F_{\tilde{\beta}}$ in $\mathscr{F}_\beta$ by reordering β into $(\gamma_n\colon 1 \le n < \alpha)$ for an $\alpha \le \omega_0$ and setting $\gamma_0 = \tilde{\beta}$. If β is a limit ordinal and if we assume that $(F_{\tilde{\beta}}\colon \tilde{\beta} < \beta)$ has already been chosen, we first observe that $I_\beta = \bigcup_{\tilde{\beta}<\beta} I_{\tilde{\beta}}$. So one can find a family $F_\beta = (x(A, B)\colon (A, B) \in I_\beta)$ such that $F_\beta|_{I_{\tilde{\beta}}} = F_{\tilde{\beta}}$ whenever $0 < \tilde{\beta} < \beta$. Since every $A \in \mathscr{P}_f(\beta)$ is already an element of $\mathscr{P}_f(\tilde{\beta})$, where $\tilde{\beta} < \beta$ us sufficiently large, F_β satisfies $(\mathscr{F}_\beta)$. □

In order to show the assumption of Lemma 6, one needs the following Lemmas 7 and 8. Lemma 7 can be shown in a similar way as [HJ, p. 3, Lemma 2], where (ACBH) is assumed, while Lemma 8 involves the classical Ramsey theorem as presented in [O, Theorem 1.1].

7. **Lemma.** *Let $(y_n'^{(i)})$ be convex blocks of (x_n'), for $i = 1, \dots, k$, $k \in \mathbf{N}$, and let $\delta > 0$. Then there exist infinite $N_1, \dots, N_k \subset \mathbf{N}$, and for every $B \subset \{1, \dots, k\}$ there exists $x(B) \in B_1(X)$ with*

$$\langle y_n'^{(i)}, x(B)\rangle - c \begin{cases} \geq (\frac{1}{2} - \delta) & \text{if } i \in B, \\ \leq -(\frac{1}{2} - \delta) & \text{if } i \notin B, \end{cases}$$
$$\text{for } i \leq k,\ n \in N_i,\ B \subset \{1, \dots, k\}.$$

Proof. By passing to subsequences if necessary, we can assume that (y_n'), where $y_n' := \frac{1}{k}\sum_{i=1}^k y_n'^{(i)}$ for $n \in \mathbf{N}$, is a convex block of (x_n') also. By Lemma 3, we find $x \in B_1(X)$ and infinite M_1, $M_2 \subset \mathbf{N}$ with

$$(7.1)\quad \langle y_n', x\rangle \geq c + \frac{1}{2} - \frac{\delta}{4k} \qquad \text{if } n \in M_1$$
$$\text{and} \quad \langle y_n', x\rangle \leq c - \frac{1}{2} + \frac{\delta}{4k} \text{ if } n \in M_2.$$

From the properties of (x_n') (compare Lemma 3), we deduce for each $i \leq k$ that

$$\begin{aligned} \limsup_{n\to\infty, n\in M_1} \langle y_n'^{(i)}, x\rangle &= \left(\limsup_{n\to\infty, n\in M_1} \langle y_n'^{(i)}, x\rangle - \liminf_{n\to\infty, n\in M_2} \langle y_n', x\rangle\right) \\ &\quad + \liminf_{n\to\infty, n\in M_2} \langle y_n', x\rangle \\ &\leq 1 + c - \frac{1}{2} + \frac{\delta}{4k} = c + \frac{1}{2} + \frac{\delta}{4k}. \end{aligned}$$

By passing to a cofinite subset of M_1, we may assume that

$$(7.2)\qquad \langle y_n'^{(i)}, x\rangle \leq c + \frac{1}{2} + \frac{\delta}{2k} \qquad \text{if } n \in M_1.$$

Similarly we prove that we may assume that $\langle y_n'^{(i)}, x\rangle \geq c - 1/2 - \delta/2k$ if $n \in M_2$. We deduce from (7.1) and (7.2) that, for each $i \leq k$ and $n \in M_1$,

$$\begin{aligned} \langle y_n'^{(i)}, x\rangle &= k\langle y_n', x\rangle - \sum_{j\leq k, j\neq i} \langle y_n'^{(j)}, x\rangle \\ &\geq k(c + 1/2 - \delta/4k) - (k-1)(c + 1/2 + \delta/2k) \\ &> c + 1/2 - \delta. \end{aligned}$$

Similarly, we deduce that $\langle y_n'^{(i)}, x\rangle < c - 1/2 + \delta$ for $i \leq k$ and $n \in M_2$. Now let $B \subset \{1, \dots, k\}$. If we define for each $i \in \{1, \dots, k\}$ $\tilde{N}_i := M_1$ if $i \in B$ and $\tilde{N}_i := M_2$ if $i \notin B$ and $x(B) := x$, then it follows for $i \leq k$ and $n \in N_i$,

$$(7.3)\qquad \langle y_n'^{(i)}, x(B)\rangle - c \begin{cases} \geq (\frac{1}{2} - \delta) & \text{if } i \in B, \\ \leq -(\frac{1}{2} - \delta) & \text{if } i \notin B. \end{cases}$$

Repeating this process for every $B \in \{B_1, \dots, B_{2^k}\} = \mathscr{P}(\{1, \dots, k\})$ we get infinite sets $\mathbf{N} \supset N_i^{(1)} \supset \cdots \supset N_i^{(2^k)}$ for every $i \leq k$ and elements $x(B_1), x(B_2), \dots, x(B_{2^k}) \in B_1(X)$ such that for every $\ell \in \{1, \dots, 2^k\}$, $i \leq k$, and $n \in N_i^{(\ell)}$, (7.3) holds for $B := B_\ell$. Taking $N_i := N_i^{(2^k)} = \bigcap_{\ell \leq 2^k} N_i^{(\ell)}$ for $i \in \{1, \dots, k\}$, we note that the assertion holds for the chosen $x(B_1), \dots, x(B_{2^k})$.

8. **Lemma.** *Let $(J_m : m \in \mathbf{N})$ be a sequence of finite sets; for every $m \in \mathbf{N}$ and $j \in J_m$ let $L_{(m,j)}$ again be a finite set. For every $m \in \mathbf{N}$, $j \in J_m$, and $\ell \in L_{(m,j)}$, let $f_{(m,j)}^{(\ell)} : \mathbf{N} + m \to \mathbf{R}$. Also, assume that $\sum_{\ell \in L_{(m,j)}} f_{(m,j)}^{(\ell)}(k) \geq 0$ for $m \in \mathbf{N}_0$, $j \in J_m$, and $k \in \mathbf{N}_0 + m$. Then there exists a subsequence (k_m) of $\mathbf{N}$, and for each $m \in \mathbf{N}$ and $j \in J_m$, a bijection $b(m,j) : \{1, 2, \dots, |L_{(m,j)}|\} \to L_{(m,j)}$, such that*

$$\sum_{\ell=1}^{|L_{(m,j)}|} f_{(m,j)}^{b(m,j)(\ell)}(k_{m_\ell}) \geq 0, \qquad \textit{whenever } m \leq m_1 < m_2 < \cdots < m_{|L_{(m,j)}|}.$$

Proof. First let $f^{(\ell)} : \mathbf{N} \to \mathbf{R}$, for $\ell \in \{1, \dots, k\}$, $k \in \mathbf{N}$, be such that $\sum_{\ell=1}^k f^{(\ell)}(n) \geq 0$ if $n \in \mathbf{N}$. We show that, for given infinite set $N \subset \mathbf{N}$, there exists an infinite $M \subset N$ and a bijection $b : \{1, \dots, k\} \to \{1, \dots, k\}$ such that

$$(8.1) \qquad \sum_{\ell=1}^{k} f^{b(\ell)}(m_\ell) \geq 0 \qquad \text{whenever } m_1 < \cdots < m_k \text{ lie in } M.$$

The classical Ramsey theorem (compare [O, Theorem 1.1 and following remarks]) states that for any infinite $\widetilde{N} \subset \mathbf{N}$ and any

$$\mathscr{A} \subset [\widetilde{N}]_k := \{(n_1, \dots, n_k) \in \widetilde{N}^k | n_1 < \cdots < n_k\}$$

there exists an infinite $\widetilde{M} \subset \widetilde{N}$ such that either $[\widetilde{M}]_k \subset \mathscr{A}$ or $\mathscr{A} \subset [\widetilde{N}]_k \backslash [\widetilde{M}]_k$. Let $\Pi = \{\pi_1, \dots, \pi_{k!}\}$ be the set of all permutations on $\{1, 2, \dots, k\}$. Setting $M^{(0)} := N$ and using Ramsey's theorem, we can choose successively for each $i \in \{1, \dots, k!\}$ an infinite $M^{(i)} \subset N$ with $M^{(i)} \subset M^{(i-1)}$ such that the set $\mathscr{A}^{\pi_i} := \{(n_1, \dots, n_k) \in [M^{(i-1)}]_k | \sum_{\ell=1}^k f^{\pi_i(\ell)}(n_\ell) \geq 0\}$ either contains $[M^{(i)}]_k$ or does not meet it. Now we have to show that there exists at least one $i \leq k!$ with $[M^{(i)}]_k \subset \mathscr{A}_{\pi_i}$. This can be seen as follows: Assuming that no $\mathscr{A}^{\pi_i}$ contains $[M^{(i)}]_k$, we conclude that $\mathscr{A}^\pi \cap [M^{(k!)}]_k = \varnothing$ for every $\pi \in \Pi$. This means that for any $m_1 < m_2 < \cdots < m_k$ in $M^{(k!)}$ and any permutation $\pi \in \Pi$, $\sum_{\ell=1}^k f^{\pi(\ell)}(m_\ell) < 0$. But this would imply, that for any $m_1 < m_2 \cdots < m_k$ of $M^{(k!)}$, $0 > \sum_{\ell=1}^k \sum_{\pi \in \Pi} f^{\pi(\ell)}(m_\ell) = (k-1)! \sum_{\ell=1}^k \sum_{j=1}^k f^{(j)}(m_\ell)$, which contradicts the assumption. Thus, we have verified the assertion stated at the beginning of the proof. Applying the same reasoning, for a fixed $m \in \mathbf{N}$ and for an infinite $N \subset \mathbf{N}_0 + m$, $|J_m|$ times, we get an infinite $M_m \subset N$ and, for every $j \in J_m$, a bijection $b(m,j): \{1, \dots, |L_{(m,j)}|\} \to L_{(m,j)}$, such that

$$\text{(8.2)} \quad \sum_{\ell=1}^{L_{(m,j)}} f_{(m,j)}^{b(m,j)(\ell)}(n_\ell) \geq 0, \qquad \text{for } j \in J_m \text{ and } n_1 < \cdots < n_{|L_{(m,j)}|} \text{ in } M_m.$$

It can be assumed that (M_m) decreases. For an increasing sequence (k_m), with $k_m \in M_m$ if $m \in \mathbf{N}$, the assertion is then satisfied.

Now we can state and show the last step of the proof of Theorem 1.

9. **Lemma.** *Suppose* $\alpha \in [1, \omega_0]$ *and that* $F = (x(A,B): (A,B) \in I_{[1,\alpha[})$ *satisfies condition* $(\mathscr{F}_{[1,\alpha[})$. *Then there exists an extension* $F_0 = (x(A,B): (A,B) \in I_{[0,\alpha[})$ *of* F, *which satisfies* $(\mathscr{F}_{[0,\alpha]})$.

Proof. By induction, we will choose for every $\beta \in [0,\alpha] \cap \omega_0$ a family $(x(A,B): A \subset \beta$, with $0 \in A$ and, if $\beta > 0$, $\beta - 1 \in A$; $B \subset 2^A)$ such that the following condition (9.1) is satisfied:

(9.1) For each $\gamma \in [\beta, \alpha] \cap \omega_0$ and $n \in \mathbf{N}$ there exists a family $(z'(\varphi, n): \varphi \in 2^\gamma)$ in X^* such that
(a) $z'(\varphi, n) \in \mathrm{co}(\{x'_m: m \geq n\})$ if $\varphi \in 2^\gamma$, and
(b)

$$\left\langle 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi, n), x(A,B) \right\rangle - c \begin{cases} \geq \Delta(A,n) & \text{if } \psi \in B, \\ \leq -\Delta(A,n) & \text{if } \psi \notin B, \end{cases}$$

whenever $A \in \mathscr{P}(\beta) \cup \mathscr{P}([1,\gamma[)$ $\psi \in 2^A$ and $B \subset 2^A$.

(Since for every $\beta \in [0,\alpha] \cap \omega_0$: $\bigcup_{0 \leq \beta' \leq \beta} \{A \subset \beta' | 0 \in A$ and, if $0 < \beta'$, $\beta' - 1 \in A\} = \{A \subset \beta | 0 \in A\}$, the value $x(A,B)$ is defined for each $A \in \mathscr{P}_f([1,\alpha[) \cup \mathscr{P}(\beta)$ and each $B \subset 2^A$ in the induction step β.)

Having done this, we get an extension $(X(A,B): A \in \mathscr{P}_f(\alpha), B \subset 2^A)$ of F satisfying $(\mathscr{F}_\alpha)$, which can be seen as follows: For an arbitrary $A \in \mathscr{P}_f(\alpha)$ and an $n \in \mathbf{N}$, one chooses $\beta \in [0,\alpha] \cap \omega_0$ with $A \subset \beta$ and a family $(z'(\varphi, n): \varphi \in 2^\beta)$ as, in (9.1). Then one observes that $(x'(\varphi, n): \varphi \in 2^A)$, can be defined by $x'(\varphi, n) := 2^{|A|-|\beta|} \sum_{\tilde{\varphi} \in 2^{\varphi,\beta}} z'(\tilde{\varphi}, n)$ for $\varphi \in 2^A$; this family satisfies (a) of $(\mathscr{F}_\alpha)$ because of (9.1)(a) and from (9.1)(b) we deduce $(\mathscr{F}_\alpha)$ (b) by the following equations:

$$\left\langle 2^{|A'|-|A|} \sum_{\varphi\in 2^{\varphi',A}} x'(\varphi,n), x(A',B')\right\rangle - c$$
$$= \left\langle 2^{|A'|-|A|} \sum_{\varphi\in 2^{\varphi',A}} 2^{|A|-|\beta|} \sum_{\tilde\varphi\in 2^{\varphi,\beta}} z'(\tilde\varphi,n), x(A',B')\right\rangle - c$$
$$= \left\langle 2^{|A'|-|\beta|} \sum_{\tilde\varphi\in 2^{\varphi,\beta}} z'(\tilde\varphi,n), x(A',B')\right\rangle - c$$
$$\begin{cases} \geq \Delta(A',n) & \text{if } \varphi' \in B', \\ \leq -\Delta(A',n) & \text{if } \varphi' \notin B', \end{cases} \quad \text{for } A' \subset A,\ \varphi\in 2^{A'} \text{ and } B' \subset 2^{A'}.$$

If $\beta = 0$, no $x(A,B)$ has to be defined. To verify (9.1), we chose for $\gamma \in [0,\alpha] \cap \omega_0$ and $n \in \mathbf{N}$ a family $(x'(\varphi,n)\colon \varphi \in 2^{[1,\gamma[}) \subset X^*$ as in $\mathscr{F}_{[1,\alpha[}$ (taking $A := [1,\gamma[)$ and set, for each $\varphi \in 2^\gamma$, $z'(\varphi,n) := x'(\varphi|_{[\alpha,\gamma[}n)$. It follows that $(z'(\varphi,n) : \varphi\in 2^\gamma)$ satisfies (a) and (b) of (9.1) for $\beta = 0$. Indeed, (9.1)(a) follows from $(\mathscr{F}_{[1,\alpha[})$, (a) and (9.1)(b) follows from $(\mathscr{F}_{[1,\alpha[})$ (b) which can be shown in the following way:

$$\left\langle 2^{|A|-|\gamma|} \sum_{\varphi\in 2^{\psi,\gamma}} z'(\varphi,n), x(A,B)\right\rangle - c$$
$$= \left\langle 2^{|A|0|[1,\gamma[|} \sum_{\varphi\in 2^{\psi,[1,\gamma[}} x'(\varphi,n), x(A,B)\right\rangle - c$$
$$\begin{cases} \geq \Delta(A,n) & \text{if } \psi\in B \\ \leq -\Delta(A,n) & \text{if } \psi \notin B, \end{cases}$$
$$\text{whenever } A \in \mathscr{P}([1,\gamma[),\ \psi\in 2^A \text{ and } B\subset 2^A.$$

Suppose now that for $\beta > 0$, $x(A,B)$ has been chosen for each $A \subset \beta - 1$ with $0 \in A$ and each $B \subset 2^A$. For $n \in \mathbf{N}$ we set $\gamma(n) := \max\{\gamma \leq \alpha\colon |\gamma| \leq n\}$ (thereby concluding that $\gamma(1) = 1, \gamma(2) = 2, \ldots$ and if $\alpha < \omega_0$, then $\alpha = \gamma(|\alpha|) = \gamma(|\alpha+1|)\ldots$); for $A \in \mathscr{P}_f(\alpha)$ we set $\ell(A) := \max(A) + 1$ (so we have $A \subset \ell(A) \subset \alpha$ for an $A \in \mathscr{P}_f(\alpha)$) and, for $\psi \in S_\alpha$, $\ell(\psi) := \ell(D(\psi))$. Choosing, for every $n \in \mathbf{N}$, $(\tilde z'(\varphi,n)\colon \varphi \in 2^{\gamma(n)\cup\beta})$ as in (9.1)(b) (for $\beta - 1$), and setting, for $\psi \in S_\alpha$ and $n\in\mathbf{N}$ with $\gamma(n) \geq \ell(\psi)$, $\tilde y'(\psi,n) = 2^{|D(\psi)|-|\gamma(n)\cup\beta|} \sum_{\varphi\in 2^{\psi,\gamma(n)\cup\beta}} \tilde z'(\varphi,n)$ we get a family $(\tilde y'(\psi,n)\colon \psi\in S_\alpha,\ \gamma(n) \geq \ell(\psi))$ with properties (9.2), (9.3) and (9.4) as stated and verified below. By (9.1)(a) (for $\beta - 1$),

(9.2) $\qquad \tilde y'(\varphi,n) \in \mathrm{co}(\{x'_m\colon m\geq n\}) \qquad \text{if } \varphi\in S_\alpha \text{ and } \gamma(n) \geq \ell(\varphi).$

From the definition of $\tilde{y}'(\psi, n)$ we have for $A \in \mathscr{P}_f(\alpha)$, $A' \subset A$, $\psi' \in 2^{A'}$ and $\gamma(n) \geq \ell(A)$

(9.3)
$$2^{|A'|-|A|} \sum_{\psi \in 2^{\psi', A}} \tilde{y}'(\psi, n) = 2^{|A'|-|A|} \sum_{\psi \in 2^{\psi', A}} 2^{|D(\psi)|-|\gamma(n)\cup\beta|} \sum_{\varphi \in 2^{\psi, \gamma(n)\cup\beta}} \tilde{z}'(\varphi, n)$$
$$= 2^{|A'|-|\gamma(n)\cup\beta|} \sum_{\varphi' \in 2^{\psi', \gamma(n)\cup\beta}} \tilde{z}'(\varphi', n) = \tilde{y}'(\psi', n).$$

Finally (9.1)(b) implies, for $A \in \mathscr{P}_f([1, \alpha[) \cup \mathscr{P}(\beta - 1)$, $\psi \in 2^A$ and $\gamma(n) \geq \ell(A)$,

(9.4)
$$\langle \tilde{y}'(\psi, n), x(A, B)\rangle - c = \left\langle 2^{|A|-|\gamma(n)\cup\beta|} \sum_{\varphi \in 2^{\psi, \gamma(n)\cup\beta}} \tilde{z}'(\varphi, n), x(A, B)\right\rangle - c$$
$$\begin{cases} \geq \Delta(A, n) & \text{if } \psi \in B \\ \leq -\Delta(A, n) & \text{if } \psi \notin B. \end{cases}$$

Define now, for $m \in \mathbf{N}$,

$$J_m := \{(A, B, \psi) | A \in \mathscr{P}_f([1, \gamma(m)]) \cup \mathscr{P}((\beta-1) \cap \gamma(m)), B \subset 2^A \text{ and } \psi \in 2^A\},$$

for $(A, B, \psi) \in J_m$, the set $L_{(m, A, B, \psi)} := 2^{\psi, A\cup\beta}$, and for $\varphi \in L_{(m, A, B, \psi)}$ and $k \geq m$:

$$f^{\varphi}_{(m, A, B, \psi)}(k) := \begin{cases} 2^{|A|-|A\cup\beta|}(\langle \tilde{y}'(\varphi, k), x(A, B)\rangle - c - \Delta(A, k)) & \text{if } \psi \in B \\ 2^{|A|-|A\cup\beta|}(-\langle \tilde{y}'(\varphi, k), x(A, B)\rangle + c - \Delta(A, k)) & \text{if } \psi \notin B. \end{cases}$$

We conclude, from (9.3) and (9.4), that the assumption of Lemma 8 is satisfied. Indeed, we have, for $m \in \mathbf{N}$, $k \geq m$ and $(A, B, \psi) \in J_m$,

$$\sum_{\varphi \in L_{(m, A, B, \psi)}} f^{\varphi}_{(m, A, B, \psi)}(k) = (2^{|A|-|A\cup\beta|} \sum_{\varphi \in 2^{\psi, A\cup B}} \pm\langle \tilde{y}'(\varphi, k), x(A, B)\rangle) \mp c - \Delta(A, k)$$
$$= \pm\langle \tilde{y}'(\psi, k), x(A, B)\rangle \mp c - \Delta(A, k) \geq 0.$$

So we can find a subsequence (k_n) of $\mathbf{N}$ such that the family $(y'(\varphi, n)\colon \varphi \in S_\alpha$, $\gamma(n) \geq \ell(\varphi))$, where $y'(\varphi, n) := \tilde{y}'(\varphi, k_n)$ if $\varphi \in S_\alpha$ and $\gamma(n) \geq \ell(\varphi)$, still satisfies (9.2), (9.3) and (9.4), and such that, moreover, the following property holds:

(9.5) For every $n \in \mathbf{N}$, $A \in \mathscr{P}([1, \gamma(n)[) \cup \mathscr{P}((\beta - 1) \cap \gamma(n))$, $B \subset 2^A$, and $\psi \in 2^A$, there exists a bijection

$$b(A, B, \psi, n) \colon \{1, \dots, 2^{|A \cup \beta| - |A|}\} \to 2^{\psi, A \cup \beta}$$

such that

$$\left\langle 2^{|A| - |A \cup \beta|} \sum_{i=1}^{2^{|A \cup \beta| - |A|}} y'(b(A, B, \psi, n)(i), n_i), x(A, B) \right\rangle - c$$
$$= \begin{cases} \displaystyle\sum_{i=1}^{2^{|A \cup \beta| - |A|}} f^{b(A, B, \psi, n)(i)}(k_{n_i}) + \Delta(A, k_n) \geq \Delta(A, n) & \\ & \text{if } \psi \in B, \\ \displaystyle -\sum_{i=1}^{2^{|A \cup \beta| - |A|}} f^{b(A, B, \psi, n)(i)}(k_{n_i}) - \Delta(A, k_n) \leq -\Delta(A, n) & \\ & \text{if } \psi \notin B, \end{cases}$$

whenever $n \leq n_1 < \cdots < n_{2^{|A \cup \beta| - |A|}}$.

By (9.2) we find an $N \in \mathscr{P}_\infty(\mathbf{N})$ such that for each $\varphi \in 2^\beta$ $(y'(\varphi, n) \colon n \in N, n > |\beta|)$ is a convex block of (x'_n). Applying Lemma 7 we find for every $\varphi \in 2^\beta$ an $N(\varphi) \in \mathscr{P}_\infty(N)$ and for every $B \subset 2^\beta$ an $x(\beta, B) \in B_1(X)$ such that

$$\text{(9.6)} \qquad \langle y'(\varphi, n), x(\beta, B) \rangle - c \begin{cases} \geq \Delta(\beta) & \text{if } \varphi \in B, \\ \leq -\Delta(\beta) & \text{if } \varphi \notin B, \end{cases}$$
$$\text{for } B \subset 2^\beta, \varphi \in 2^\beta \text{ and } n \in N(\varphi).$$

For an arbitrary $A \subset \beta$ with 0, $(\beta - 1) \in A$ and $B \subset 2^A$ we set $x(A, B) := x(\beta, \bigcup_{\psi \in B} 2^{\psi, \beta})$.

Now we have to verify (9.1). Toward this end let $n \in \mathbf{N}$ and $\gamma \in [\beta, \alpha] \cap \omega_0$ be arbitrary. We may assume that $\gamma(n) \geq \gamma$, otherwise we replace n by a sufficiently large $\tilde{n} \in \mathbf{N}$. We choose $\ell \in \mathbf{N}$ such that

$$\ell \geq 12 \cdot n \cdot 2^{2|\beta|} \cdot \sup_{j \in \mathbf{N}}(\|x'_j\| + 1).$$

Next we choose for each $i \in \{1, \dots, \ell\}$ and $\varphi \in 2^\beta$ an $n(\varphi, i) \in \mathbf{N}$ with
(9.7)
(a) $n(\varphi, i) \geq 2n$ and $n(\varphi, i) \in N(\varphi)$,
(b) $\max(\{n(\varphi, i-1) | \varphi \in 2^\beta\}) < \min(\{n(\varphi, i) | \varphi \in 2^\beta\})$, if $1 < i \leq \ell$.

By (9.7)(a) and (9.2), the family $(z'(\varphi, n)\colon \varphi \in 2^{\gamma})$ satisfies (9.1)(a). To show (9.1)(b), let $A \in \mathscr{P}([1, \gamma[) \cup \mathscr{P}(\beta)$, $B \subset 2^A$, and $\psi \in 2^A$; it remains to show

$$(9.8) \qquad \langle 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi, n), x(A, B)\rangle - c \begin{cases} \geq \Delta(A, n) & \text{if } \psi \in B, \\ \leq -\Delta(A, n) & \text{if } \psi \notin B. \end{cases}$$

To do this, we consider two cases:

Case 1. $0 \in A$ and $(\beta - 1) \in A$ (thus $A \subset \beta$ and $x(A, B)$ was defined in the present induction step). For this case we remark first that, by (9.3),

$$2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi, n) = \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A|-|\beta|} \sum_{\varphi' \in 2^{\psi,\beta}} y'(\varphi', n(\varphi', i)).$$

Moreover, $\psi \in B \Leftrightarrow 2^{\psi,\beta} \subset \bigcup_{\tilde{\psi} \in B} 2^{\tilde{\psi},\beta}$ and $\psi \notin B \Leftrightarrow 2^{\psi,\beta} \cap \bigcup_{\tilde{\psi} \in B} 2^{\tilde{\psi},\beta} = \emptyset$; thus, by the definition of $x(A, B)$

$$\begin{aligned} &\langle 2^{|A|-|\beta|} \sum_{\varphi' \in 2^{\psi,\beta}} y'(\varphi, n(\varphi', i)), x(A, B)\rangle - c \\ &= 2^{|A|-|\beta|} \sum_{\varphi' \in 2^{\psi,\beta}} \left\langle y'(\varphi', n(\varphi', i)), x\left(A, \bigcup_{\tilde{\psi} \in B} 2^{\tilde{\psi}}\right)\right\rangle - c \\ &\begin{cases} \geq \frac{1}{2}(1 - \frac{1}{|\beta|+1}) \geq \Delta(A, n) & \text{if } \psi \in B, \\ \leq -\frac{1}{2}(1 - \frac{1}{|\beta|+1}) \leq -\Delta(A, n) & \text{if } \psi \notin B, \end{cases} \end{aligned}$$

which implies (9.8).

Case 2. $A \in \mathscr{P}(\beta - 1) \cup \mathscr{P}([1, \gamma[)$ (thus, $x(A, B)$ was chosen in a previous induction step or was given by the assumption). Setting $b := b(A, B, \psi, 2n)$ (compare (9.5) and remark that

$$A \in \mathscr{P}_f([1, \gamma[) \cup \mathscr{P}_f(\beta - 1) \subset \mathscr{P}_f([1, \gamma(2n)[) \cup \mathscr{P}_f(\beta - 1))$$

we obtain

$$
\begin{aligned}
2^{|A|-|\gamma|} & \sum_{\varphi\in 2^{\psi,\gamma}} z'(\varphi, n) \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell} 2^{|A|-|\gamma|} \sum_{\varphi\in 2^{\psi,\gamma}} y'(\varphi, n(\varphi|_\beta, i) \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell} 2^{|A|-|A\cup\beta|} \sum_{\varphi'\in 2^{\psi,A\cup\beta}} 2^{|A\cup\beta|-|\gamma|} \sum_{\varphi''\in 2^{\varphi',\gamma}} y'(\varphi'', n(\varphi'|_\beta, i)) \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell} 2^{|A|-|A\cup\beta|} \sum_{\varphi'\in 2^{\psi,A\cup\beta}} y'(\varphi', n(\varphi'|_\beta, i)) \qquad \text{[by (9.3)]} \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell} 2^{|A\cup\beta|-|A|} \sum_{j=1}^{2^{|A\cup\beta|-|A|}} y'(b(j), n(b(j)|_\beta, i) \\
& \qquad\qquad \text{[the image of } b \text{ is } 2^{\psi,A\cup\beta}] \\
&= \frac{1}{\ell}2^{|A\cup\beta|-|A|} \\
&\cdot\Bigg[\sum_{i=1}^{\ell+1-2^{|A|-|A\cup\beta|}} \sum_{j=1}^{2^{|A|-|A\cup\beta|}} y'(b(j), n(b(j)|_\beta, i-1+j)) \\
&\quad + \sum_{i=1}^{2^{|A|-|A\cup\beta|}} \sum_{j=i+1}^{2^{|A|-|A\cup B|}} y'(b(j), n(b(j)|_\beta, i)) \\
&\quad + \sum_{i=\ell+2-2^{|A|-|A\cup\beta|}}^{\ell} \sum_{j=1}^{i-(\ell+1-2^{|A|-|A\cup\beta|})} y'(b(j), n(b(j)|_\beta, i))\Bigg] \\
& \qquad\qquad \text{[by changing the order of summation].}
\end{aligned}
$$

Now we remark that the norm of the second and third sum between the brackets of the last lines does not exceed the value $2^{2|\beta|}\sup_{j\in\mathbf{N}}\|x_j'\|$, which is not greater than $\ell/12n$ by the choice of ℓ. For the first sum, we remark that by (9.7)(b)

$$2n \le n(b(1), i-1+1) < n(b(2), i-1+2)\cdots < n(b(2^{|A\cup\beta|-|A|}), i-1+2^{|A\cup\beta|-|A|}),$$

whenever $i\in\{1, \dots, l+1-2^{|A\cup\beta|-|A|}\}$. It follows from (9.5) that the first sum multiplied with $1/\ell(2^{A\cup\beta|-|A|})$ is, up to the factor $q := \ell/(\ell+1-2^{|A\cup\beta|-|\beta|})$ a convex combination of elements y' which fulfill

$$\langle y', x(A, B)\rangle - c\begin{cases} \ge \Delta(A, 2n) & \text{if } \psi\in B, \\ \le -\Delta(A, 2n) & \text{if } \psi\notin B, \end{cases}$$

From the choice of ℓ it follows that $|1-q|\le 1/12n$ which implies the assertion (9.8) and finishes the proof.

3. An application to the limited sets in Banach spaces

A subset A of a Banach space X is said to be *limited* if all weak*-convergent sequences in X^* converge uniformly on A. It is easy to see that all relatively compact sets are limited, while in [BD] it was shown that every limited set has to be weakly conditionally compact. More about limited sets can be found in [BD, DE, S].

In [BD, Proposition 7] it was shown that in Banach spaces, not containing ℓ_1, every limited set is relatively weakly compact. This was done by proving first that spaces possessing limited sets which are not relatively weakly compact enjoy property (CBH).

With Corollary 2 we get the following generalization of this result (remark that by [P], $L_1(\{0,1\}^{\mathbf{N}}) \subset X^*$ iff $\ell_1 \subset X$):

10. **Corollary.** *If the dual of a Banach space X does not contain $L_1(\{0,1\}^{\omega_p})$, then all limited sets are relatively weakly compact.*

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