ON DUAL SPACES WITH BOUNDED SEQUENCES WITHOUT WEAK * CONVERGENT CONVEX BLOCKS

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Abstract. In this work we show that if $X^*$ contains bounded sequences without weak* convergent convex blocks, then it contains an isometric copy of $L_1(\{0,1\}^{\omega_1})$.

1. Introduction

We are concerned with the relation between properties of the weak* topology of the dual $X^*$ of a Banach space $X$ and the property of $X$ containing $\ell_1(\Gamma)$, or of $X^*$ containing $L_1(\{0,1\}^{\Gamma})$ for a set $\Gamma$. The results of this manuscript are related to those of J. Bourgain [B], R. Haydon [Hy], R. Haydon, M. Levy and E. Odell [HLO] and J. Hagler and W. B. Johnson [HJ]; in particular, they generalize results obtained in [B, Hy, HJ].

The notations and terminology are mostly standard. The first infinite ordinal is denoted by $\omega_0$; the first uncountable by $\omega_1$ and the first ordinal with the cardinality of the continuum, by $\omega_c$. The ordinal $\omega_p$ is taken to be the smallest ordinal such that there exists a family $(N_\xi)_{\xi<\omega_p}$ of infinite subsets of $\mathbb{N}$ having the property that $\bigcap_{\xi \in F} N_\xi$ is infinite for every finite $F \subset \omega_p$, but not admitting an infinite $N \subset \mathbb{N}$, such that $N \setminus N_\xi$ is finite for each $\xi < \omega_p$. It is easy to see that, $\omega_1 \leq \omega_p \leq \omega_c$. More about $\omega_p$ can be found in [F]; it is known for example, that $\omega_1 < \omega_p = \omega_c$ if we assume $\neg$CH and MA by their definition $\omega_0, \omega_1, \omega_p, \omega_c$ are initial ordinals and can so be identified with cardinals. Only for technical reasons do we distinguish between the finite ordinals and the elements of the positive integers $\mathbb{N}$, which we consider as cardinals.

For a set $\Gamma$, the cardinality is denoted by $|\Gamma|$; and $\mathcal{P}_f(\Gamma)$ and $\mathcal{P}_\infty(\Gamma)$ denote the set of all finite and infinite subsets of $\Gamma$, whereas $\mathcal{P}(\Gamma)$ denotes the power set. For simplicity, we consider only Banach spaces over the real field $\mathbb{R}$; for a Banach space $X$, $B_1(X)$ shall mean the unit ball and $X^*$, the dual

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space of $X$. The weak topology on $X$ and the weak*-topology on $X^*$ are also denoted by $\sigma(X, X^*)$ and $\sigma(X^*, X)$ respectively.

For a set $\Gamma$, $L_1(\{0,1\}^\Gamma)$ is the $L_1$-space for the product measure

$$\bigotimes_{\gamma \in \Gamma} \frac{1}{2}(\delta_0 + \delta_1)$$

on the set $\{0,1\}^\Gamma$ furnished with the product $\sigma$-algebra $\bigotimes_{\gamma \in \Gamma} \mathcal{P}(\{0,1\})$. We consider the following two properties of a Banach space $X$ concerning the weak* topology on $X^*$:

We say that the Banach space $X$ satisfies

(CBH) (convex block hypothesis) if $X^*$ contains a bounded sequence $(x'_n)$ which has no $\sigma(X^*, X)$-convergent convex block, and

(ACBH) (absolutely convex block hypothesis) if $X^*$ contains a bounded sequence $(x'_n)$ which has no $\sigma(X^*, X)$-convergent absolutely convex block basis,

where a sequence of the form $(\sum_{i=k_n}^{k_{n+1}-1} a_i x'_i : n \in \mathbb{N})$ is called a convex block (respectively an absolutely convex block basis) of $(x'_n)$ if $(k_n)$ is increasing in $\mathbb{N}$, $(a_n) \subset \mathbb{R}_0^+$ (respectively $(a_n) \subset \mathbb{R}$), and $\sum_{i=k_n}^{k_{n+1}-1} a_i = 1$ (respectively $\sum_{i=k_n}^{k_{n+1}-1} |a_i| = 1$) for each $n \in \mathbb{N}$.

It is obvious that (ACBH) implies (CBH) and we remark that (ACBH) is equivalent to the condition, considered by J. Hagler and W. B. Johnson [HJ] and by R. Haydon [Hy], that $X^*$ contains an infinite-dimensional subspace $Y$ in which $\sigma(X^*, X)$-convergence of sequences implies norm convergence. In [HJ] it was first observed that nonreflexive Grothendieck spaces enjoy (ACBH) and it was proven that (ACBH) implies that $X$ contains an isometric copy of $\ell_1$. R. Haydon [Hy] improved this result by showing that (ACBH) implies that $L_1(\{0,1\}^\omega)$ is isometrically embedded in $X^*$. J. Bourgain and J. Diestel showed in [BD] that spaces having limited sets [cf. §3] which are not relatively weakly compact have the property (CBH) and in [B] it was shown that (CBH) implies that $X$ contains an isometric copy of $\ell_1$. Finally it was proven in [HLO] that under the (set-theoretical) assumption that $\omega_1 < \omega_p$ (CBH) implies that $X$ contains a copy of $\ell_1(\omega_p)$, which is under this hypothesis equivalent to $L_1(\{0,1\}^{\omega_p}) \subset X^*$ [ABZ]; the nonreflexive Grothendieck space constructed in [T] under CH does not contain any copy of $\ell_1(\omega_1)$ and, thus, shows that the result in [HLO] is dependent on further set-axioms.

Our main purpose is to show:

1. Theorem. If $X$ has property (CBH), then $X^*$ contains an isometric copy of $L_1(\{0,1\}^{\omega_1})$.

Together with the above-cited result of [HLO] we deduce:

2. Corollary. If $X$ satisfies property (CBH), then $X^*$ contains an isometric copy of $L_1(\{0,1\}^{\omega_p})$. 

2. Proof of Theorem 1

The following lemma is due to H. P. Rosenthal [R]:

3. Lemma (cited from [HLO, p. 4, Lemma 3A]). Let \( X \) satisfy \((CBH)\). Then there exists a bounded sequence \((x'_n)\) in \( X^* \) and \( c \in \mathbb{R} \) such that for every convex block \((y'_n)\) of \((x'_n)\) and every \( \eta < \frac{1}{2} \) there exists \( x \in B_1(X) \) such that

\[
\limsup_{n \to \infty} \langle y'_n, x \rangle > c + \eta, \quad \liminf_{n \to \infty} \langle y'_n, x \rangle < c - \eta,
\]

and

\[
\sup_{\hat{x} \in B_1(X)} \left| \limsup_{n \to \infty} \langle x'_n, \hat{x} \rangle - \liminf_{n \to \infty} \langle x'_n, \hat{x} \rangle \right| = 1.
\]

For the sequel, we assume that \( X \) has property \((CBH)\) and that we have chosen \((x'_n) \subset X^* \) and \( c \in \mathbb{R} \) as in Lemma 3. To handle the space \( L_1(\{0,1\}^\Gamma) \) for a nonempty set \( \Gamma \), we need the following notations: For a set \( A \), the set of all mappings \( \varphi: A \to \{0,1\} \) will be denoted by \( 2^A \); for \( A' \subset A \) and \( \varphi' \in 2^{A'} \), the set of all extensions of \( \varphi' \) onto the whole of \( A \) will be denoted by \( 2^{\varphi', A} \). The union \( \bigcup \{2^A|A \in \mathcal{P}_f(\Gamma)\} \) is denoted by \( S_\Gamma \) and for the domain of \( \varphi \in S_\Gamma \) we write \( D(\varphi) \).

R. Haydon [Hy, p. 6, Lemma 3] provided the following characterization for a Banach space \( Y \) to contain an isometric copy of \( L_1(\{0,1\}^\Gamma) \).

4. Lemma. Let \( Y \) be a Banach space and \( \Gamma \) a set. Then \( Y \) contains an isometric copy of \( L_1(\{0,1\}^\Gamma) \) if and only if there exists a family \((y_\varphi: \varphi \in S_\Gamma)\) in \( Y \) satisfying (a) and (b) as given below:

(a) \( y_\varphi = 2^{\varphi' - |A'|} \sum_{\varphi' \in 2^{\varphi', A}} y_{\varphi'} \) for any \( A \in \mathcal{P}_f(\Gamma), A' \subset A \) and \( \varphi' \in 2^{A'} \)

(since \( |2^{\varphi', A}| = 2^{|A'|-|A'|} \), this means that \( y_\varphi \) is the arithmetic mean of \((y_{\varphi'}: \varphi \in 2^{\varphi', A})\)).

(b) \( \left| \sum_{\varphi \in 2^A} a_\varphi y_\varphi \right| = \sum_{\varphi \in 2^A} |a_\varphi| \) for any \( A \in \mathcal{P}_f(\Gamma) \) and \((a_\varphi: \varphi \in 2^A) \subset \mathbb{R} \).

In this case, there is an isometry \( T: L_1(\{0,1\}^\Gamma) \to Y \) such that \( T(e_\varphi) = y_\varphi \) for \( \varphi \in S_\Gamma \), where \( e_\varphi \in L_1(\{0,1\}^\Gamma) \) is defined by

\[
e_\varphi := 2^{|D(\varphi)|} \chi_{\{\theta \in 2^\Gamma|\theta(\gamma) = \varphi(\gamma) \text{ if } \gamma \in D(\varphi)\}}.
\]

Another sufficient condition, for \( X^* \) to contain \( L_1(\{0,1\}^\Gamma) \) can be formulated using the following definition.
Definition. Let $\Gamma$ be a set. A family $F = (x(A, B) : A \in \mathcal{P}(\Gamma), B \subset 2^A)$ in $B_1(X)$ is said to satisfy $(\mathcal{T}_F)$ if the following condition holds:

$$(\mathcal{T}_F) \quad \text{For every } A \in \mathcal{P}(\Gamma) \text{ and } n \in \mathbb{N} \text{ there exists a family }$$
$$x'(\varphi, n) : \varphi \in 2^A \text{ such that }$$

(a) $x'(\varphi, n) \in co\{x'_m | m \geq n\}$, if $\varphi \in 2^A$, and
(b) $2^{\lfloor |A'| - |A| \rfloor} \sum_{\varphi \in 2^{A'}} x'(\varphi, n), x(A', B') \geq c \begin{cases} \frac{1}{2}(1 = \frac{1}{|A'| + 1} - \frac{1}{n}) & \text{if } \varphi' \in B', \\ -\frac{1}{2}(\frac{1}{|A'| + 1} - \frac{1}{n}) & \text{if } \varphi' \notin B' \end{cases}$

whenever $A' \subset A$, $\varphi' \in 2^{A'}$ and $B' \subset 2^{A'}$. For the sake of brevity, we will denote the set $\{(A, B) | A \in \mathcal{P}(\Gamma), B \subset 2^A\}$ by $I_\Gamma$, the set of all families $F = (x(A, B) : (A, B) \in I_\Gamma)$ which satisfy $(\mathcal{T}_F)$ by $\mathcal{F}_\Gamma$; and the values $\frac{1}{2}(1 - 1/(|A| + 1) - 1/n)$ and $\frac{1}{2}(1 - 1/(|A| + 1))$ by $\Delta(A, n)$ and $\Delta(A)$ respectively for $A \in \mathcal{P}(\Gamma)$ and $n \in \mathbb{N}$.

With these definitions we are in a position to state the following result.

5. Lemma. Let $\Gamma$ be an infinite set. If $\mathcal{F}_\Gamma \neq \emptyset$, then there exists an isometric copy of $L_1(\{0, 1\}^\Gamma)$ in $X^*$.

Proof. Let $F = (x(A, B) : (A, B) \in I_\Gamma) \subset B_1(X)$ satisfy property $(\mathcal{T}_F)$. For each $\varphi \in S_\Gamma$ and each $n \in \mathbb{N}$ choose $x'(\varphi, n) \in B_x(X^*)$ as prescribed in $(\mathcal{T}_F)$ and define for each $\psi \in S_\Gamma$ and each $A \in \mathcal{P}(\Gamma)$

$$(5.1) \quad y'(\psi, A) := 2^{\lfloor D(\psi)\cap A - |A| \rfloor} \sum_{\varphi \in 2^{D(\psi)\cap A}} x'(\varphi, |A| + 1).$$

The net $(y'(\psi, A) : \psi \in S_\Gamma, A \in \mathcal{P}(\Gamma))$ has an accumulation point $(y'(\psi) : \psi \in S_\Gamma)$ in the product $K := \prod_{\psi \in S_\Gamma} co\{x'_n | n \in \mathbb{N}\}^\omega^\ast$, endowed with the product of the weak* topology on $co\{x'_n : n \in \mathbb{N}\}^\omega^\ast$ (the elements of $\mathcal{P}(\Gamma)$ are ordered by inclusion). From $(\mathcal{T}_F)$ and (5.1), it follows that $(y'(\psi) : \psi \in S_\Gamma)$ fulfills the following three properties (5.2), (5.3) and (5.4):

$$(5.2) \quad y'(\psi) \in \bigcap_{n \in \mathbb{N}} co\{x'_m : m \geq n\}^\omega^\ast \text{ for each } \psi \in S_\Gamma,$$

$$(5.3) \quad y'(\psi') = 2^{\lfloor A' - |A| \rfloor} \sum_{\varphi \in 2^{A'} \setminus A} y'(\psi), \quad \text{for } A' \subset A \in \mathcal{P}(\Gamma) \text{ and } \psi' \in 2^{A'},$$

$$(5.4) \quad (y'(\psi), x(A, B)) - c \begin{cases} \geq \Delta(A) & \text{if } \psi \in B, \\ \leq -\Delta(A) & \text{if } \psi \notin B, \\ \end{cases} \text{ for } A \in \mathcal{P}(\Gamma), \psi \in 2^A \text{ and } B \subset 2^A$$
[Since $y'(\psi)$ is a $w^*$-accumulation-point of the net $(y'(\psi, \tilde{A}) : \tilde{A} \in \mathcal{P}_f(\Gamma))$, with $D(\psi) \subset \tilde{A}$.]

We now choose a fixed $\gamma \in \Gamma$. Since $\Gamma$ is infinite, it suffices to show that the family $(y'(\psi^1) - y'(\psi^0)) : \psi \in S_{\Gamma \setminus \{\gamma\}}$, satisfies (a) and (b) of Lemma 4, where for $\theta \in \{0, 1\}$, and $\psi \in S_{\Gamma \setminus \{\gamma\}}$, $\psi^\theta \in 2^{D(\psi) \cup \{\gamma\}}$, is given by $\psi^\theta |_{D(\psi)} = \psi$ and $\psi^\theta(\gamma) = \theta$. Condition (a) follows from (5.3). In order to show (b), let $A \in \mathcal{P}_f(\Gamma \setminus \{\gamma\})$ and $(a^\phi : \phi \in 2^A) \subset \mathbb{R}$. From (5.2) and Lemma 3 it follows that for any $x \in B_1(X)$ and $\phi \in 2^A$ we have $(x, y'(\phi^1) - y'(\phi^0)) \leq 1$, which implies that $\| \sum_{\phi \in 2^A} a_\phi (y'(\phi^1) - y'(\phi^0)) \| \leq \sum_{\phi \in 2^A} |a_\phi|$. To show $\geq$, let $\varepsilon > 0$. Without loss of generality, assume $2\Delta(A) \geq 1 - \varepsilon$. Otherwise replace $A$ by an $\tilde{A} \in \mathcal{P}_f(\Gamma \setminus \{\gamma\})$ with $A \subset \tilde{A}$ and $2\Delta(\tilde{A}) \geq 1 - \varepsilon$ and note that by (5.3) we have

$$\sum_{\phi \in 2^A} 2^{1 - |\tilde{A}|} a_\phi(\phi^1 - \phi^0) = \sum_{\phi \in 2^A} a_\phi(\phi^1 - \phi^0).$$

Now take $x := x(A \cup \{\gamma\}, \{\phi^1 | \phi \in 2^A \text{ and } a_\phi \geq 0\} \cup \{\phi^0 | \phi \in 2^A \text{ and } a_\phi < 0\})$. By (5.4) we have

$$\left\| \sum_{\phi \in 2^A} a_\phi(y'(\phi^1) - y'(\phi^0)) \right\| \geq \sum_{\phi \in 2^A} a_\phi(x, y'(\phi^1) - y'(\phi^0)) \geq \sum_{\phi \in 2^A} a_\phi \text{sign}(a_\phi) 2\Delta(A) \geq (1 - \varepsilon) \sum_{\phi \in 2^A} |a_\phi|.$$

The assertion follows since $\varepsilon \geq 0$ was arbitrary.

By Lemma 5, it is enough to show that $\mathcal{F}_{\omega_1} \neq \emptyset$. As we will see from Lemma 6, it is sufficient to show that for every $\alpha \in [1, \omega_0]$ each $F \in \mathcal{F}_{[1, \alpha]}$ can be extended to an $F_0 \in \mathcal{F}_{[0, \alpha]}$.

6. **Lemma.** Suppose that for every $\alpha \in [1, \omega_0]$, each family $F = (x(A, B), (A, B) \in I_{[1, \alpha]} \subset B_1(X)$ satisfying $(\mathcal{F}_{[1, \alpha]})$ can be extended to an $F_0 = (x(A, B), (A, B) \in I_{[0, \alpha]}$ which satisfies $(\mathcal{F}_{[0, \alpha]}).$ Then $\mathcal{F}_{\omega_1}$ is not empty; in particular, $L_1((0, 1)^{(\omega_1)})$ can be embedded in $X^*$.

**Proof.** In order to show that there exists an $F \in \mathcal{F}_{\omega_1}$, we define an $F_\beta \in \mathcal{F}_\beta$ by transfinite induction for every $\beta \in [0, \omega_1]$ such that $F_\beta |_{I_\beta} = F_\beta$ whenever $\dot{\beta} < \beta$. If $\beta = \dot{\beta} + 1$, with $\dot{\beta} < \omega_1$ and with $F_\beta \in \mathcal{F}_\beta$ having been chosen, one can use the assumption to get an extension $F_\beta$ of $F_\beta$ in $\mathcal{F}_\beta$ by reordering $\beta$ into $(\gamma_n : 1 \leq n < \alpha)$ for an $\alpha \leq \omega_0$ and setting $\gamma_0 = \dot{\beta}$. If $\beta$ is a limit ordinal and if we assume that $(F_\beta : \dot{\beta} < \beta)$ has already been chosen, we first observe that $I_\beta = \bigcup_{\dot{\beta} < \beta} I_\beta$. So one can find a family $F_\beta = (x(A, B), (A, B) \in I_\beta)$ such that $F_\beta |_{I_\beta} = F_\beta$ whenever $0 < \dot{\beta} < \beta$. Since every $A \in \mathcal{P}_f(\beta)$ is already an element of $\mathcal{P}_f(\tilde{\beta})$, where $\tilde{\beta} < \beta$ us sufficiently large, $F_\beta$ satisfies $(\mathcal{F}_{\tilde{\beta}})$. □
In order to show the assumption of Lemma 6, one needs the following Lemmas 7 and 8. Lemma 7 can be shown in a similar way as [HJ, p. 3, Lemma 2], where (ACBH) is assumed, while Lemma 8 involves the classical Ramsey theorem as presented in [O, Theorem 1.1].

7. Lemma. Let \((y'^{(i)}_n)\) be convex blocks of \((x'_n)\), for \(i = 1, \ldots, k\), \(k \in \mathbb{N}\), and let \(\delta > 0\). Then there exist infinite \(N_1, \ldots, N_k \subset \mathbb{N}\), and for every \(B \subset \{1, \ldots, k\}\) there exists \(x(B) \in B_1(X)\) with
\[
(y'^{(i)}_n, x(B)) = c \begin{cases} 
(\frac{1}{2} - \delta) & \text{if } i \in B, \\
-(\frac{1}{2} - \delta) & \text{if } i \notin B,
\end{cases}
\]
for \(i \leq k\), \(n \in N_i\), \(B \subset \{1, \ldots, k\}\).

Proof. By passing to subsequences if necessary, we can assume that \((y'_n)\), where \(y'_n := \frac{1}{k} \sum_{i=1}^{k} y'^{(i)}_n\) for \(n \in \mathbb{N}\), is a convex block of \((x'_n)\) also. By Lemma 3, we find \(x \in B_1(X)\) and infinite \(M_1, M_2 \subset \mathbb{N}\) with
\[
(y'_n, x) \geq c + \frac{1}{2} - \frac{\delta}{4k} \quad \text{if } n \in M_1
\]
and
\[
(y'_n, x) \leq c - \frac{1}{2} + \frac{\delta}{4k} \quad \text{if } n \in M_2.
\]
From the properties of \((x'_n)\) (compare Lemma 3), we deduce for each \(i \leq k\) that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{n \in M_1} (y'^{(i)}_n, x) = \left( \limsup_{n \to \infty} \frac{1}{n} \sum_{n \in M_1} (y'^{(i)}_n, x) - \liminf_{n \to \infty} \frac{1}{n} \sum_{n \in M_2} (y'_n, x) \right) + \liminf_{n \to \infty} \frac{1}{n} \sum_{n \in M_2} (y'_n, x)
\leq 1 + c - \frac{1}{2} + \frac{\delta}{4k} = c + \frac{1}{2} + \frac{\delta}{4k}.
\]
By passing to a cofinite subset of \(M_1\), we may assume that
\[
\langle y'^{(i)}_n, x \rangle \leq c + \frac{1}{2} + \frac{\delta}{2k} \quad \text{if } n \in M_1.
\]
Similarly we prove that we may assume that \(\langle y'^{(i)}_n, x \rangle \geq c - 1/2 - \delta/2k\) if \(n \in M_2\). We deduce from (7.1) and (7.2) that, for each \(i \leq k\) and \(n \in M_1\),
\[
\langle y'^{(i)}_n, x \rangle = k(y'_n, x) - \sum_{j \leq k, j \neq i} \langle y'^{(j)}_n, x \rangle
\geq k(c + 1/2 - \delta/4k) - (k-1)(c + 1/2 + \delta/2k)
> c + 1/2 - \delta.
\]
Similarly, we deduce that \(\langle y'^{(i)}_n, x \rangle < c - 1/2 + \delta\) for \(i \leq k\) and \(n \in M_2\). Now let \(B \subset \{1, \ldots, k\}\). If we define for each \(i \in \{1, \ldots, k\}\) \(\tilde{N}_i := M_1\) if \(i \in B\) and \(\tilde{N}_i := M_2\) if \(i \notin B\) and \(x(B) := x\), then it follows for \(i \leq k\) and \(n \in N_i\),
\[
\langle y'^{(i)}_n, x(B) \rangle = c \begin{cases} 
(\frac{1}{2} - \delta) & \text{if } i \in B, \\
-(\frac{1}{2} - \delta) & \text{if } i \notin B.
\end{cases}
\]
Repeating this process for every \( B \in \{B_1, \ldots, B_{2k}\} = \mathcal{P}\{1, \ldots, k\} \) we get infinite sets \( N \supset N_i^{(1)} \supset \cdots \supset N_i^{(2^k)} \) for every \( i \leq k \) and elements \( x(B_1), x(B_2), \ldots, x(B_{2k}) \in B_i(X) \) such that for every \( \ell \in \{1, \ldots, 2^k\}, i \leq k, \) and \( n \in N_i^{(\ell)}, (7.3) \) holds for \( B := B_\ell \). Taking \( N_i := N_i^{(2^k)} = \bigcap_{\ell \leq 2^k} N_i^{(\ell)} \) for \( i \in \{1, \ldots, k\} \), we note that the assertion holds for the chosen \( x(B_1), \ldots, x(B_{2k}) \).

8. **Lemma.** Let \((J_m : m \in \mathbb{N})\) be a sequence of finite sets; for every \( m \in \mathbb{N} \) and \( j \in J_m \) let \( L(m, j) \) again be a finite set. For every \( m \in \mathbb{N} \), \( j \in J_m \), and \( \ell \in L(m, j) \), let \( f^{(\ell)}(J_m, j) : \mathbb{N} + m \to \mathbb{R} \). Also, assume that \( \sum_{\ell \in L(m, j)} f^{(\ell)}(j, m) (k) \geq 0 \) for \( m \in \mathbb{N}_0, j \in J_m, \) and \( k \in \mathbb{N}_0 + m \). Then there exists a subsequence \( (k_m) \) of \( N \), and for each \( m \in \mathbb{N} \) and \( j \in J_m \), a bijection \( b(m, j) : \{1, \ldots, |L(m, j)|\} \to L(m, j) \) such that

\[
\sum_{\ell=1}^{|L(m, j)|} f^{b(m, j)}(m, j) (k_m) \geq 0, \quad \text{whenever } m \leq m_1 < m_2 < \cdots < m_{|L(m, j)|}.
\]

**Proof.** First let \( f^{(\ell)} : \mathbb{N} \to \mathbb{R}, \) for \( \ell \in \{1, \ldots, k\}, k \in \mathbb{N} \), be such that

\[
\sum_{\ell=1}^k f^{(\ell)}(n) \geq 0 \text{ if } n \in \mathbb{N}.
\]

We show that, for given infinite set \( N \subset \mathbb{N} \), there exists an infinite \( M \subset N \) and a bijection \( b : \{1, \ldots, k\} \to \{1, \ldots, k\} \) such that

\[
\sum_{\ell=1}^k f^{b(\ell)}(m_\ell) \geq 0 \quad \text{whenever } m_1 < \cdots < m_k \text{ lie in } M.
\]

The classical Ramsey theorem (compare [O, Theorem 1.1 and following remarks]) states that for any infinite \( \tilde{N} \subset \mathbb{N} \) and any

\[
\mathcal{A} \subset [\tilde{N}]_k := \{(n_1, \ldots, n_k) \in \tilde{N}^k : n_1 < \cdots < n_k\}
\]

there exists an infinite \( \tilde{M} \subset \tilde{N} \) such that either \( [\tilde{M}]_k \subset \mathcal{A} \) or \( \mathcal{A} \subset [\tilde{N}]_k \setminus[\tilde{M}]_k \). Let \( \Pi = \{\pi_1, \ldots, \pi_k!\} \) be the set of all permutations on \( \{1, 2, \ldots, k\} \). Setting \( M^{(0)} := N \) and using Ramsey’s theorem, we can choose successively for each \( i \in \{1, \ldots, k!\} \) an infinite \( M^{(i)} \subset N \) with \( M^{(i)} \subset M^{(i-1)} \) such that the set \( \mathcal{A}^{\pi_i} := \{(n_1, \ldots, n_k) \in [M^{(i-1)}]_k : \sum_{\ell=1}^k f^{\pi_i(\ell)}(n_\ell) \geq 0\} \) either contains \( [M^{(i)}]_k \) or does not meet it. Now we have to show that there exists at least one \( i \leq k! \) with \( [M^{(i)}]_k \subset \mathcal{A}^{\pi_i} \). This can be seen as follows: Assuming that no \( \mathcal{A}^{\pi_i} \) contains \( [M^{(i)}]_k \), we conclude that \( \mathcal{A}^{\pi_i} \cap [M^{(k!)}]_k = \emptyset \) for every \( i \in \Pi \). This means that for any \( m_1 < m_2 < \cdots < m_k \) in \( M^{(k!)} \) and any permutation \( \pi \in \Pi, \) \( \sum_{\ell=1}^k f^{\pi(\ell)}(m_\ell) < 0 \). But this would imply, that for any \( m_1 < m_2 < \cdots < m_k \) of \( M^{(k!)} \), \( 0 > \sum_{\ell=1}^k \sum_{\pi \in \Pi} f^{\pi(\ell)}(m_\ell) = (k-1)! \sum_{\ell=1}^k \sum_{j=1}^k f^{j(\ell)}(m_\ell) \), which contradicts the assumption. Thus, we have verified the assertion stated at the beginning of the proof. Applying the same reasoning, for a fixed \( m \in \mathbb{N} \) and for an infinite \( N \subset \mathbb{N}_0 + m, |J_m| \) times, we get an infinite \( M_m \subset N \) and, for
every \( j \in J_m \), a bijection \( b(m, j): \{1, \ldots, |L_{(m, j)}|\} \rightarrow L_{(m, j)} \), such that

\[
\sum_{\ell=1}^{L_{(m, j)}} f_{(m, j)}^{b(m, j)(\ell)}(n_\ell) \geq 0, \quad \text{for } j \in J_m \text{ and } n_1 < \cdots < n_{|L_{(m, j)}|} \text{ in } M_m.
\]

It can be assumed that \( (M_m) \) decreases. For an increasing sequence \( (k_m) \), with \( k_m \in M_m \) if \( m \in \mathbb{N} \), the assertion is then satisfied.

Now we can state and show the last step of the proof of Theorem 1.

9. Lemma. Suppose \( \alpha \in [1, \omega_0] \) and that \( F = (x(A, B): (A, B) \in I_{[1, \alpha]}^0) \) satisfies condition \( (\mathcal{T}_{[1, \alpha]}^0) \). Then there exists an extension \( F_0 = (x(A, B): (A, B) \in I_{[0, \alpha]}^0) \) of \( F \), which satisfies \( (\mathcal{T}_{[0, \alpha]}^0) \).

Proof. By induction, we will choose for every \( \beta \in [0, \alpha] \cap \omega_0 \) a family \( (x(A, B): A \subseteq B, 0 \subseteq A \text{ and, if } \beta > 0, \beta - 1 \subseteq A; B \subseteq 2^A) \) such that the following condition (9.1) is satisfied:

\[
(9.1) \quad \text{For each } \gamma \in [\beta, \alpha] \cap \omega_0 \text{ and } n \in \mathbb{N} \text{ there exists a family } \\
(z'(\varphi, n): \varphi \in 2^\gamma) \text{ in } X^* \text{ such that } \\
(a) \quad z'(\varphi, n) \in \text{co}(\{x_m: m \geq n\}) \text{ if } \varphi \in 2^\gamma, \text{ and } \\
(b) \quad \\
\left\{ \begin{array}{ll}
2^{[A]-[\gamma]} \sum_{\varphi \in 2^\gamma} z'(\varphi, n), x(A, B) & -c \\
\geq \Delta(A, n) & \text{if } \psi \in B, \\
\leq -\Delta(A, n) & \text{if } \psi \notin B,
\end{array} \right.
\]

whenever \( A \in \mathcal{P}(\beta) \cup \mathcal{P}([1, \gamma]) \). \( \psi \in 2^A \) and \( B \subseteq 2^A \).

(Since for every \( \beta \in [0, \alpha] \cap \omega_0: \bigcup_{0 \leq \beta' \leq \beta} \{A \subseteq B|0 \subseteq A \text{ and, if } 0 \leq \beta', \beta' - 1 \subseteq A\} = \{A \subseteq B|0 \subseteq A\}, \text{ the value } x(A, B) \text{ is defined for each } A \in \mathcal{P}_f([1, \alpha]) \cup \mathcal{P}(\beta) \text{ and each } B \subseteq 2^A \text{ in the induction step } \beta. \)

Having done this, we get an extension \( (X(A, B): A \in \mathcal{P}_f(\alpha), B \subseteq 2^A) \) of \( F \) satisfying \( (\mathcal{T}_\alpha^0) \), which can be seen as follows: For an arbitrary \( A \in \mathcal{P}_f(\alpha) \) and an \( n \in \mathbb{N} \), one chooses \( \beta \in [0, \alpha] \cap \omega_0 \) with \( A \subseteq \beta \) and a family \( (z'(\varphi, n): \varphi \in 2^\beta) \) as, in (9.1). Then one observes that \( (x'(\varphi, n): \varphi \in 2^A) \), can be defined by \( x'(\varphi, n) := 2^{[A]-[\beta]} \sum_{\varphi \in 2^\gamma} z'(\varphi, n) \) for \( \varphi \in 2^A \); this family satisfies (a) of \( (\mathcal{T}_\alpha^0) \) because of (9.1)(a) and from (9.1)(b) we deduce \( (\mathcal{T}_\alpha^0) \) (b)
by the following equations:

\[
\left\langle 2^{|A'|-|A|} \sum_{\varphi \in 2^{*}} x'(\varphi, n), x(A', B') \right\rangle - c
\]
\[
= \left\langle 2^{|A'|-|A|} \sum_{\varphi \in 2^{*}} 2^{|A|} \sum_{\varphi' \in 2^{*}} z'(\varphi, n), x(A', B') \right\rangle - c
\]
\[
= \left\langle 2^{|A'|-|\beta|} \sum_{\varphi \in 2^{*}} z'(\varphi, n), x(A', B') \right\rangle - c
\]
\[
\begin{cases}
\geq \Delta(A', n) & \text{if } \varphi' \in B', \\
\leq -\Delta(A', n) & \text{if } \varphi' \notin B',
\end{cases}
\]

If \( \beta = 0 \), no \( x(A, B) \) has to be defined. To verify (9.1), we chose for \( \gamma \in [0, \alpha] \cap \omega_0 \) and \( n \in \mathbb{N} \) a family \( (x'(\varphi, n): \varphi \in 2^{[1, \alpha]} \subset X^* \) as in \( \mathcal{F}_{[1, \alpha]} \) (taking \( A := [1, \gamma] \)) and set, for each \( \varphi \in 2^\gamma \), \( z'(\varphi, n) := x'(\varphi |_{[1, \gamma]}, n) \). It follows that \( (z'(\varphi, n): \varphi \in 2^\gamma) \) satisfies (a) and (b) of (9.1) for \( \beta = 0 \). Indeed, (9.1)(a) follows from \( (\mathcal{F}_{[1, \alpha]})(a) \) and (9.1)(b) follows from \( (\mathcal{F}_{[1, \alpha]})(b) \) which can be shown in the following way:

\[
\left\langle 2^{|A'|-|\gamma|} \sum_{\varphi \in 2^{\gamma}} z'(\varphi, n), x(A, B) \right\rangle - c
\]
\[
= \left\langle 2^{|A|0[1, \alpha]} \sum_{\varphi \in 2^{\alpha}} x'(\varphi, n), x(A, B) \right\rangle - c
\]
\[
\begin{cases}
\geq \Delta(A, n) & \text{if } \psi \in B \\
\leq -\Delta(A, n) & \text{if } \psi \notin B,
\end{cases}
\]
whenever \( A \in \mathcal{P}([1, \gamma]), \psi \in 2^A \) and \( B \subset 2^A \).

Suppose now that for \( \beta > 0 \), \( x(A, B) \) has been chosen for each \( A \subset \beta - 1 \) with \( 0 \in A \) and each \( B \subset 2^A \). For \( n \in \mathbb{N} \) we set \( \gamma(n) := \max\{\gamma \leq \alpha: |\gamma| \leq n\} \) (thereby concluding that \( \gamma(1) = 1, \gamma(2) = 2, \ldots \) and if \( \alpha < \omega_0 \), then \( \alpha = \gamma(|\alpha|) = \gamma(|\alpha + 1|) \ldots \) ) for \( A \in \mathcal{P}_f(\alpha) \) we set \( \ell(A) := \max(A) + 1 \) (so we have \( A \subset \ell(A) \subset \alpha \) for an \( A \in \mathcal{P}_f(\alpha) \) ) and, for \( \psi \in S_\alpha \), \( \ell(\psi) := \ell(D(\psi)) \). Choosing, for every \( n \in \mathbb{N} \), \( (z'(\varphi, n): \varphi \in 2^{[\gamma(n) \cup \beta]} \) as in (9.1)(b) (for \( \beta - 1 \) ), and setting, for \( \psi \in S_\alpha \) and \( n \in \mathbb{N} \) with \( \gamma(n) \geq \ell(\psi) \), \( \tilde{y}'(\psi, n) = 2^{|D(\psi)|-|\gamma(n) \cup \beta|} \sum_{\varphi \in 2^{\gamma(n) \cup \beta}} z'(\varphi, n) \) we get a family \( (\tilde{y}'(\psi, n): \psi \in S_\alpha, \gamma(n) \geq \ell(\psi)) \) with properties (9.2), (9.3) and (9.4) as stated and verified below. By (9.1)(a) (for \( \beta - 1 \)),

\[
(9.2) \quad \tilde{y}'(\varphi, n) \in \text{co}\{x'_m: m \geq n\} \quad \text{if } \varphi \in S_\alpha \text{ and } \gamma(n) \geq \ell(\varphi).
\]
From the definition of $y'(\psi, n)$ we have for $A \in \mathcal{P}_f(\alpha)$, $A' \subset A$, $\psi' \in 2^{A'}$ and $\gamma(n) \geq \ell(A)$

\[(9.3)\]

$$2^{|A'|-|A|} \sum_{\psi \in 2^{A'} \setminus A} y'(\psi, n) = 2^{|A'|-|A|} \sum_{\psi \in 2^{A'} \setminus A} 2^{|D(\psi)|-|\gamma(n) \cup B|} \sum_{\varphi \in 2^{\mathcal{V}, \gamma(n) \cup B}} z'(\varphi, n) = 2^{|A'|-|\gamma(n) \cup B|} \sum_{\varphi \in 2^{A', \gamma(n) \cup B}} z'(\varphi, n) = y'(\psi', n).$$

Finally (9.1)(b) implies, for $A \in \mathcal{P}_f([1, \alpha]) \cup \mathcal{P}(\beta - 1)$, $\psi \in 2^{A'}$ and $\gamma(n) \geq \ell(A)$,

\[(9.4)\]

$$\langle y'(\psi, n), x(A, B) \rangle - c = \left(2^{|A'|-|\gamma(n) \cup B|} \sum_{\varphi \in 2^{\mathcal{V}, \gamma(n) \cup B}} z'(\varphi, n), x(A, B) \right) - c$$

\[\begin{cases}
\geq \Delta(A, n) & \text{if } \psi \in B \\
\leq -\Delta(A, n) & \text{if } \psi \notin B.
\end{cases}\]

Define now, for $m \in \mathbb{N}$,

$$J_m := \{(A, B, \psi) | A \in \mathcal{P}_f([1, \gamma(m)]) \cup \mathcal{P}((\beta - 1) \cap \gamma(m)), B \subset 2^{A'} \text{ and } \psi \in 2^{A'}\},$$

for $(A, B, \psi) \in J_m$, the set $L_{(m,A,B,\psi)} := 2^{\mathcal{V}, A \cup B}$, and for $\varphi \in L_{(m,A,B,\psi)}$ and $k \geq m$:

$$f^{\varphi}_{(m,A,B,\psi)}(k) := \begin{cases}
2^{|A'|-|A \cup B|} (\langle y'(\varphi, k), x(A, B) \rangle - c - \Delta(A, k)) & \text{if } \psi \in B \\
2^{|A'|-|A \cup B|} (-\langle y'(\varphi, k), x(A, B) \rangle + c - \Delta(A, k)) & \text{if } \psi \notin B.
\end{cases}$$

We conclude, from (9.3) and (9.4), that the assumption of Lemma 8 is satisfied. Indeed, we have, for $m \in \mathbb{N}$, $k \geq m$ and $(A, B, \psi) \in J_m$,

$$\sum_{\varphi \in L_{(m,A,B,\psi)}} f^{\varphi}_{(m,A,B,\psi)}(k) = (2^{|A'|-|A \cup B|} \sum_{\varphi \in 2^{\mathcal{V}, A \cup B}} \pm \langle y'(\varphi, k), x(A, B) \rangle) = c - \Delta(A, k)$$

$$= \pm \langle y'(\psi, k), x(A, B) \rangle = c - \Delta(A, k) \geq 0.$$ 

So we can find a subsequence $(k_n)$ of $\mathbb{N}$ such that the family $(y'(\varphi, n) : \varphi \in S_\alpha, \gamma(n) \geq \ell(\varphi))$, where $y'(\varphi, n) := y'(\varphi, k_n)$ if $\varphi \in S_\alpha$ and $\gamma(n) \geq \ell(\varphi)$, still satisfies (9.2), (9.3) and (9.4), and such that, moreover, the following property
holds:

\[(9.5) \text{ For every } n \in \mathbb{N}, A \in \mathcal{P}(\{1, \gamma(n)\}) \cup \mathcal{P}((\beta - 1) \cap \gamma(n)), B \subset 2^A, \text{ and } \psi \in 2^A, \text{ there exists a bijection } \]
\[b(A, B, \psi, n): \{1, \ldots, 2^{\lvert A \cup B \rvert - \lvert A \rvert} \} \to 2^\psi \cup A \cup B \]
\[\text{ such that } \]
\[\left\langle 2^{\lvert A \cup B \rvert - \lvert A \rvert} \sum_{i=1}^{2^{\lvert A \cup B \rvert - \lvert A \rvert}} y'(b(A, B, \psi, n)(i), n), x(A, B) \right\rangle - c \]
\[= \begin{cases} 
\sum_{i=1}^{2^{\lvert A \cup B \rvert - \lvert A \rvert}} f^b(A, B, \psi, n)(i)(k_{n_i}) + \Delta(A, k_{n_i}) \geq \Delta(A, n) \\
- \sum_{i=1}^{2^{\lvert A \cup B \rvert - \lvert A \rvert}} f^b(A, B, \psi, n)(i)(k_{n_i}) - \Delta(A, k_{n_i}) \leq -\Delta(A, n) 
\end{cases}
\text{ if } \psi \in B, \\
\text{ if } \psi \notin B, \\
\text{ whenever } n \leq n_1 < \cdots < n_{2^{\lvert A \cup B \rvert - \lvert A \rvert}}. \]

By (9.2) we find an \( N \in \mathcal{P}_\infty(N) \) such that for each \( \varphi \in 2^\beta \) \((y'(\varphi, n): n \in N, n > \lvert \beta \rvert)\) is a convex block of \((x'_n)\). Applying Lemma 7 we find for every \( \varphi \in 2^\beta \) an \( N(\varphi) \in \mathcal{P}_\infty(N) \) and for every \( B \subset 2^\beta \) an \( x(\beta, B) \in B_1(X) \) such that

\[(9.6) \langle y'(\varphi, n), x(\beta, B) \rangle - c \begin{cases} 
\geq \Delta(\beta) \quad \text{ if } \varphi \in B, \\
\leq -\Delta(\beta) \quad \text{ if } \varphi \notin B, 
\end{cases}
\text{ for } B \subset 2^\beta, \varphi \in 2^\beta, \text{ and } n \in N(\varphi). \]

For an arbitrary \( A \subset \beta \) with \( 0, (\beta - 1) \in A \) and \( B \subset 2^A \) we set \( x(A, B) := x(\beta, \bigcup_{\psi \in B} 2^\psi \cup A) \).

Now we have to verify (9.1). Toward this end let \( n \in \mathbb{N} \) and \( \gamma \in [\beta, \alpha] \cap \omega_0 \) be arbitrary. We may assume that \( \gamma(n) \geq \gamma \), otherwise we replace \( n \) by a sufficiently large \( h \in \mathbb{N} \). We choose \( \ell \in \mathbb{N} \) such that

\[\ell \geq 12 \cdot n \cdot 2^{2^\beta} \cdot \sup_{j \in \mathbb{N}}(\lVert x'_j \rVert + 1).\]

Next we choose for each \( i \in \{1, \ldots, \ell\} \) and \( \varphi \in 2^\beta \) an \( n(\varphi, i) \in \mathbb{N} \) with

\[(9.7) \begin{align*}
(a) \quad & n(\varphi, i) \geq 2n \quad \text{and} \quad n(\varphi, i) \in N(\varphi), \\
(b) \quad & \max(\{n(\varphi, i - 1) \mid \varphi \in 2^\beta\}) < \min(\{n(\varphi, i) \mid \varphi \in 2^\beta\}), \quad \text{if } 1 < i \leq \ell.
\end{align*}\]
By (9.7)(a) and (9.2), the family \((z'(\varphi, n) : \varphi \in 2^\gamma)\) satisfies (9.1)(a). To show (9.1)(b), let \(A \in \mathcal{P}([1, \gamma]) \cup \mathcal{P}(\beta), \ B \subset 2^A, \) and \(\psi \in 2^A; \) it remains to show

\[
\langle 2^{|A| - |\gamma|} \sum_{\varphi \in 2^\gamma} z'(\varphi, n), x(A, B) \rangle - c \left\{ \begin{array}{ll}
\geq \Delta(A, n) & \text{if } \psi \in B, \\
\leq -\Delta(A, n) & \text{if } \psi \notin B.
\end{array} \right.
\]

(9.8)

To do this, we consider two cases:

**Case 1.** \(0 \in A \) and \((\beta - 1) \in A \) (thus \(A \subset \beta \) and \(x(A, B)\) was defined in the present induction step). For this case we remark first that, by (9.3),

\[
2^{|A| - |\gamma|} \sum_{\varphi \in 2^\gamma} z'(\varphi, n) = \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A| - |\beta|} \sum_{\varphi' \in 2^\gamma} y'(\varphi', n(\varphi', i)).
\]

Moreover, \(\psi \in B \Leftrightarrow 2^{\psi, \beta} \subset \bigcup_{\varphi \in B} 2^{\varphi, \beta} \) and \(\psi \notin B \Leftrightarrow 2^{\psi, \beta} \cap \bigcup_{\varphi \in B} 2^{\varphi, \beta} = \emptyset; \) thus, by the definition of \(x(A, B)\)

\[
\langle 2^{|A| - |\beta|} \sum_{\varphi' \in 2^\gamma} y'(\varphi', n(\varphi', i)), x(A, B) \rangle - c
\]

\[
= 2^{|A| - |\beta|} \sum_{\varphi' \in 2^\gamma} \left\langle y'(\varphi', n(\varphi', i)), x(A, \bigcup_{\varphi \in B} 2^{\varphi}) \right\rangle - c
\]

\[
\left\{ \begin{array}{ll}
\geq \frac{1}{2}(1 - \frac{1}{|\beta| + 1}) \geq \Delta(A, n) & \text{if } \psi \in B, \\
\leq -\frac{1}{2}(1 - \frac{1}{|\beta| + 1}) \leq -\Delta(A, n) & \text{if } \psi \notin B,
\end{array} \right.
\]

which implies (9.8).

**Case 2.** \(A \in \mathcal{P}(\beta - 1) \cup \mathcal{P}([1, \gamma]) \) (thus, \(x(A, B)\) was chosen in a previous induction step or was given by the assumption). Setting \(b := b(A, B, \psi, 2n)\) (compare (9.5) and remark that

\[
A \in \mathcal{P}_{f}(\{1, \gamma\}) \cup \mathcal{P}_{f}(\beta - 1) \subset \mathcal{P}_{f}([1, \gamma(2n)]) \cup \mathcal{P}_{f}(\beta - 1))
\]
we obtain
\[2^{|A|-|\gamma|} \sum_{\varphi \in 2^\nu \setminus A} z'_{\varphi, n} \]
\[= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A|-|\gamma|} \sum_{\varphi \in 2^\nu \setminus A} y'_{\varphi, n(\varphi |, i)} \]
\[= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A|-|A \cup \beta|} \sum_{\varphi' \in 2^\nu \setminus A \cup \beta} 2^{|A \cup \beta|-|\gamma|} \sum_{\varphi'' \in 2^\nu \setminus A \cup \beta} y'_{\varphi'' , n(\varphi' |, i)} \]
\[= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A|-|A \cup \beta|} \sum_{\varphi' \in 2^\nu \setminus A \cup \beta} y'_{\varphi' , n(\varphi' |, i)} \quad \text{[by (9.3)]} \]
\[= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A \cup \beta|-|\gamma|} \sum_{j=1}^{2^{|A \cup \beta|-|\gamma|}} y'(b(j), n(b(j)|, i)) \]
\[= \frac{1}{\ell} 2^{|A \cup \beta|-|\gamma|} \sum_{i=1}^{\ell+1-2^{|A|-|A \cup \beta|}} \sum_{j=1}^{2^{|A|-|A \cup \beta|}} y'(b(j), n(b(j)|, i-1+j)) \]
\[+ \sum_{i=1}^{2^{|A|-|A \cup \beta|}} \sum_{j=i+1}^{2^{|A|-|A \cup \beta|}} y'(b(j), n(b(j)|, i)) \]
\[+ \sum_{i=\ell+2-2^{|A|-|A \cup \beta|}}^{\ell+1-2^{|A|-|A \cup \beta|}} \sum_{j=1}^{2^{|A|-|A \cup \beta|}} y'(b(j), n(b(j)|, i)) \]
\[\text{[by changing the order of summation].} \]

Now we remark that the norm of the second and third sum between the brackets of the last lines does not exceed the value \(2^{|\beta|} \sup_{j \in \mathbb{N}} \|x'_j\|\), which is not greater than \(\ell/12n\) by the choice of \(\ell\). For the first sum, we remark that by (9.7)(b)
\[2n \leq n(b(1), i-1+1) < n(b(2), i-1+2) \cdots < n(b(2^{|A \cup \beta|-|A|}, i-1+2^{|A \cup \beta|-|A|}), \]
whenever \(i \in \{1, \ldots, \ell+1-2^{|A \cup \beta|-|A|}\}\). It follows from (9.5) that the first sum multiplied with \(1/\ell(2^{|A \cup \beta|-|A|})\) is, up to the factor \(q := \ell/(\ell+1-2^{|A \cup \beta|-|\beta|})\) a convex combination of elements \(y'\) which fulfill
\[\langle y', x(A,B) \rangle - c \begin{cases} \geq \Delta(A, 2n) & \text{if } \psi \in B, \\ \leq -\Delta(A, 2n) & \text{if } \psi \notin B, \end{cases} \]
From the choice of \(\ell\) it follows that \(|1-q| \leq 1/12n\) which implies the assertion (9.8) and finishes the proof.
3. AN APPLICATION TO THE LIMITED SETS IN BANACH SPACES

A subset $A$ of a Banach space $X$ is said to be limited if all weak*-convergent sequences in $X^*$ converge uniformly on $A$. It is easy to see that all relatively compact sets are limited, while in [BD] it was shown that every limited set has to be weakly conditionally compact. More about limited sets can be found in [BD, DE, S].

In [BD, Proposition 7] it was shown that in Banach spaces, not containing $\ell_1$, every limited set is relatively weakly compact. This was done by proving first that spaces possessing limited sets which are not relatively weakly compact enjoy property (CBH).

With Corollary 2 we get the following generalization of this result (remark that by [P], $L_1([0,1]^\mathbb{N}) \subset X^*$ iff $\ell_1 \subset X$):

10. Corollary. If the dual of a Banach space $X$ does not contain $L_1([0,1]^\mathbb{N})$, then all limited sets are relatively weakly compact.

References


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