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ON THE EXISTENCE OF IDEMPOTENT LIFTINGS

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ABSTRACT. An existence theorem for idempotent liftings is proved. This implies that every compact measure space with full support and separable measure algebra admits an idempotent lifting.

1. INTRODUCTION AND REVIEW OF TERMINOLOGY

The problems treated in this note derive from the viewpoint of lifting theory developed in the paper of P. Georgiou [5]. More precisely, we are concerned with the existence of idempotent liftings for a compact measure space.

The classical bibliography yields many positive results on the strong lifting property and Losert's example [9] shows that the existence of strong liftings is not always possible. However (except the known results concerning strong liftings), relatively little is known about the existence of idempotent liftings.

In this paper, using classical arguments of disintegration theory, we establish an existence theorem for idempotent liftings. As a consequence, we have that—without any set theoretical assumptions—a separable measure space admits always an idempotent lifting. We note, in passing, that Martin's Axiom (and the continuum hypothesis) implies also the strong lifting property (cf. [3, 11(b), p. 8]) but—in ZFC alone—the existence of a strong lifting, for every compact separable measure space, might be an indecidable problem, for all that is known.

1.1. Let T be a compact space, μ a positive Radon measure on T with $\operatorname{supp}\mu = T$, $M^{\infty}(T,\mu)$ the algebra of bounded measurable real valued functions on T, $L^{\infty}(T,\mu)$ the Banach algebra $M^{\infty}(T,\mu)$ modulo negligible functions and $h \to \tilde{h}$ the canonical projection of $M^{\infty}(T,\mu)$ onto $L^{\infty}(T,\mu)$. For a lifting r of $M^{\infty}(T,\mu)$ the function $\psi_r\colon T \to T$ is defined by: $rf = f \circ \psi_r$, $f \in C(T)$ (cf. [5]).

1.2. Let $\Lambda(T,\mu)$ be the set of all liftings for (T,μ). We define the operation $*\colon \Lambda(T,\mu) \times \Lambda(T,\mu) \to \Lambda(T,\mu)\colon (r_1, r_2) \to r_1 * r_2\colon (r_1 * r_2)f := (r_1 f) \circ \psi_{r_2}$,

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$f \in M^{\infty}(T, \mu)$. Then, $(\Lambda(T, \mu), *)$ becomes a semigroup whose right unit elements are exactly the strong liftings for (T, μ) (cf. [5]).

A lifting r for (T, μ) is called idempotent, if $r * r = r$. Clearly, r is idempotent iff $rf = (rf) \circ \psi_r$ for all $f \in M^{\infty}(T, \mu)$.

1.3. Let X, T be compact spaces, λ, μ positive Radon measures on X, T resp. and $p: X \to T$ a λ-proper mapping, with $p(\lambda) = \mu$. Then, a family $(\nu_t)_{t \in T}$ of measures on X will be said to constitute a p-disintegration of λ [8], if for every function $f \geq 0$ on X measurable, the following conditions are satisfied:

(1) f is ν_t-measurable a.e. (μ);

(2) the (μ-almost everywhere defined) function $t \to \nu_t(f)$ is μ-measurable and for every subset A of T μ-measurable, we have

$$\int_{p^{-1}A} f \, d\lambda = \int_A \nu_t(f) \, d\mu(t).$$

2. The lifting theorem

Throughout this section X, T are compact spaces, λ, μ positive Radon measures supported on X, T resp. and $p: X \to T$ a λ-proper surjection with $p(\lambda) = \mu$.

Denote by p_* the mapping: $L^{\infty}(T, \mu) \to L^{\infty}(X, \lambda): \tilde{g} \to g\tilde{\circ}p$.

We start proving a slight extension of a theorem by A. and C. Ionescu Tulcea [8, Ch. IX, p. 138, Th. 3].

2.1. **Lemma.** *Let r be a lifting for (T, μ). Then, there is a p-disintegration $(\nu_t)_{t \in T}$ of λ with the properties*:

(i) $\sup \nu_t \subset p^{-1}\{\psi_r(t)\}$, *a.e.* (μ),

(ii) $\langle f, \nu \rangle \leq r\langle f, \nu \rangle$, $f \in C_+(X)$ (*where for a disintegration* $(\mu_t)_{t \in T}$ *and a universally measurable function* h *on* X, $\langle h, \mu \rangle$ *is the function* $t \to \mu_t(h)$).

Proof. This is similar to that of [8, Ch. IX, p. 154, Th. 5]. We sketch the steps of the proof.

I. For $f \in C(X)$ define the measure ν_f on T by

$$\nu_f(g) := \lambda(f(g \circ p)), \qquad g \in C(T). \tag{3}$$

We immediately verify that ν_f is absolutely continuous with respect to μ and by Radon-Nikodym's theorem [2], we find a function $\kappa_f \in M^{\infty}(T, \mu)$ such that

$$\nu_f(A) = \int_A \kappa_f(t) \, d\mu(t), \qquad \text{for all } \mu\text{-measurable sets } A. \tag{4}$$

For $t \in T$, define the measure κ_t on $X: \kappa_t(f) := r\kappa_f(t)$, $f \in C(X)$.

II. We shall prove that $(\kappa_t)_{t\in T}$ is a p-disintegration of λ. We easily verify that (3) holds for every nonnegative μ-measurable function g and so,

$$\lambda(f\cdot\chi_{p^{-1}A}) = \int_A \kappa_t(f)\,d\mu(t) = \int \kappa_t(f)d(\chi_A\mu)(t) \tag{5}$$

(where χ_C denotes the characteristic function of a set C and $\chi_A\mu$ the measure on T: $(\chi_A\mu)(g) := \mu(\chi_A g)$, $g\in C(T)$).

Applying now [8, Ch. III, p. 40, Th. 3], to the family of functions $(r\kappa_f)_{f\in C_+(X)}$, we show that (5) holds for f lower semi-continuous and (by the regularity of λ) for every $f\in M^\infty(X,\lambda)$. This means that $(\kappa_t)_{t\in T}$ is a p-disintegration of λ.

III. Since p is λ-proper, there exists a disjoint sequence $\{K_n\}$ of compact sets in X, such that $\lambda(X-\bigcup_n K_n) = 0$ and each $p_n = p|K_n : K_n \to T$ is continuous. For $n\in\mathbf{N}$ and $t\in T$ consider the measure $\nu_{n,t}$ on X: $\nu_{n,t}(f) := r\langle f\chi_{K_n},\kappa\rangle(t)$, $f\in C(X)$. Then, since $\sum_n \nu_t(f)\le \|f\|_\infty$, $f\in C(X)$, $t\in T$, we (can) define on X the measure $\nu_t := \sum_n \nu_{n,t}$.

We claim that $(\nu_t)_{t\in T}$ is the desired p-disintegration. For this, we verify that

A. For every $f\in C_+(X)$, $\nu_t(f) = \kappa_t(f)$, a.e. (μ). Let A be any μ-measurable set. Then, by [8, Ch. III, p. 40, Th. 3], we have that $\int_A \nu_t(f)\,d\mu(t) = \int_A \sum_n r\langle f\chi_{K_n},\kappa\rangle(t)\,d\mu(t) = \sum_n \int_A \kappa_t(f\chi_{K_n})\,d\mu(t) = \sum_n \lambda(f\chi_{K_n}\chi_{p^{-1}A}) = \lambda(f\chi_{p^{-1}A})$ and using also [8, Ch. III, p. 40, Th. 3] that

B. $\lambda(f\chi_{p^{-1}A}) = \int_A \nu_t(f)\,d\mu(t)$, for every $f\in M^\infty(X,\lambda)$ and A μ-measurable.

It remains to check that (i) is true ((ii) is obvious). For this, it is sufficient to verify that

(#) If A is a compact subset of X, $g\in C(X)$ with $g|_{A^c} = 0$ and $\psi_r(t)\notin p(A)$, then $\nu_t(g) = 0$.

In fact, under these notations,

$$\begin{aligned}\nu_{n,t}(g) &= r\langle g\chi_{K_n},\kappa\rangle(t) \le \|g\|_\infty r\langle \chi_{A\cap K_n},\kappa\rangle\\ &\le \|g\|_\infty \chi_{rp(A\cap K_n)}(t) \qquad \text{(because } \kappa_t(A\cap K_n)\le \chi_{p(A\cap K_n)}(t) \text{ a.e. } (\mu))\\ &\le \|g\|_\infty \chi_{\psi_r^{-1}p(A\cap K_n)}(t) \qquad \text{(because } p(A\cap K_n) \text{ is compact)}\\ &\le \|g\|_\infty \chi_{\psi_r^{-1}p(A)}(t)\end{aligned}$$

and (#) follows.

2.2. **Theorem.** *Suppose that p_* is onto $L^\infty(X,\lambda)$ and there is an idempotent lifting for (T,μ). Then there is an idempotent lifting for (X,λ).*

Proof. We shall use the notations in the proof of Lemma 2.1. Suppose that r is idempotent.

Claim 1. For every $t\in T$, κ_t is a Dirac measure.

Since $(\kappa_t)_{t\in T}$ is a p-disintegration of λ and $r\langle 1,\kappa\rangle = \langle 1,\kappa\rangle$, κ_t is a probability measure, for all t. We show that each κ_t is a multiplicative functional (on $C(X)$). Indeed, take $f_i \in C(X)$, $i = 1,2$. By assumption, there is $g_i \in M^\infty(T,\mu)$ such that $f_i = g_i \circ p$ a.e. (μ). Then, for every μ-mesurable set A, $\int_A \kappa_t(f_i)\,d\mu(t) = \int_{p^{-1}A} f_i\,d\lambda = \int_{p^{-1}A}(g_i\circ p)\,d\lambda = \int_A g_i\,d\mu$, so,

$$\langle f_i,\kappa\rangle = g_i \quad \text{a.e. } (\mu) \text{ (and similarly)}, \tag{6}$$

$$\langle f_1f_2,\kappa\rangle = g_1g_2 \quad \text{a.e. } (\mu). \tag{7}$$

Thus, $r\langle f_1f_2,\kappa\rangle = \langle f_1f_2,\kappa\rangle = r(g_1g_2) = rg_1rg_2$ and Claim 1 follows.

Now, let $q\colon T\to X$ be the mapping defined by: $\kappa_t = \delta_{q(t)}$, $t\in T$, where δ_x is the Dirac measure at $x\in X$. Since $(\kappa_t)_{t\in T}$ is a p-disintegration of λ, q satisfies the condition:

$$(q\circ p)^{-1}A \underset{\lambda}{\equiv} A, \qquad \text{for all } \lambda\text{-measurable sets } A. \tag{+}$$

Claim 2. μ-almost for all $t\in T$, we have $p\circ q(t) = \psi_r(t)$.

Since $\nu_t(1) = 1$ a.e. (μ) and $\nu_t \le \delta_{q(t)}$ for all $t\in T$, μ-almost for all t, $\kappa_t = \nu_t$ and Claim 2 follows combining Claim 1 and Lemma 2.1.

Define now the lifting u of $M^\infty(X,\lambda)$ by: $uh := r(h\circ q)\circ p$, $h\in M^\infty(X,\lambda)$. u is in fact a lifting: the main task is to show that $uh \equiv_\lambda h$. But, this is obvious, since, by $(+)$, $r(h\circ q)\circ p \equiv_\lambda h\circ q\circ p \equiv_\lambda h$. Moreover, for $h\in C(X)$ and $x\in X$, $uh(x) = r(h\circ q)\circ p(x) = r\langle h,\kappa\rangle\circ p(x) = \langle h,\kappa\rangle\circ p(x) = \kappa_{p(x)}(h) = h\circ q\circ p(x)$. This means that $\psi_u = q\circ p$.

Consider the sets

$$E := \{t\in T\colon \psi_r(t) = p\circ q(t)\}$$

and

$$F := \{x\in X\colon uh(x) = (uh)\circ\psi_u(x)\}.$$

Since $\psi_u = q\circ p$, we deduce $(uh)\circ\psi_u(x) = uh(x)$, for all $h\in M^\infty(X,\lambda)$ and $x\in p^{-1}E$. Thus, $\lambda(F) = \lambda(X)$.

Define now the mapping $v\colon L^\infty(X,\lambda)\to M^\infty(X,\lambda)$:

$$\tilde h \to v\tilde h\colon v\tilde h(x) := \begin{cases} u\tilde h(x), & x\in F,\\ \phi_x\tilde h, & x\notin F,\end{cases} \tag{8}$$

where ϕ_x is a character of $L^\infty(X,\lambda)$ such that $\phi_x\tilde f = f(x)$, $f\in C(X)$ (cf. [8, Ch. V]). It is easily seen that v is a lifting for (X,λ). We claim that v is in fact idempotent. (8) shows that

$$\psi_v(x) = \begin{cases} \psi_u(x), & x\in F,\\ x, & x\notin F.\end{cases}$$

Take any $h\in M^\infty(X,\lambda)$ and $x\in X$. Then

$$(vh)\circ\psi_v(x) = \begin{cases} vh(\psi_u(x)), & x\in F,\\ vh(x), & x\notin F,\end{cases} = \begin{cases} uh(\psi_u(x)), & x\in F,\\ vh(x), & x\notin F,\end{cases} = vh(x).$$

So, v is idempotent.

2.3. **Corollary.** *Let* (X, λ) *be a compact (probability) measure space with* $\sup \lambda = X$ *and separable measure algebra. Then, there exists an idempotent lifting for* (X, λ).

Proof. W.l.o.g. we can take λ to be nonatomic. Let μ be the Lebesgue measure on $[0, 1]$. By [10], there is a measure preserving surjection $p: X \to [0, 1]$. Since $[0, 1]$ is compact and metrizable, p is λ-proper (cf. [2, Ch. IV, p. 179, Th. 4]). On the other hand, p_* is onto (cf. [2]) and there is a strong lifting for μ. The conclusion follows from Theorem 2.2.

2.4. **Remarks and Examples.** (a) Let μ be the Lebesgue measure on $T = [0, 1]$, (X, λ) the hyperstonian space associated with (T, μ), p the canonical mapping: $X \to T$ and r an arbitrary strong lifting for (T, μ). Then (under the notations in the proof of 2.2, one can easily verify that) v provides an example of an idempotent lifting, which is not almost strong. (However, it is not known if—in general—the existence of idempotent liftings implies the strong lifting property).

(b) The special case of Theorem 2.2, where p is continuous and (T, μ) admits a strong lifting, has been proved by P. Georgiou [6], using the "special disintegration" (cf. [4]).

(c) Suppose that there is an idempotent lifting for (X, ν). Then, every measure (supported on X and) absolutely continuous with respect to ν admits an idempotent lifting (this has been proved in [7] and extends a theorem of K. Bichteler—cf. [1]).

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