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NEW COMBINATORIAL INTERPRETATIONS OF TWO ANALYTIC IDENTITIES

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ABSTRACT. Two generalized partition theorems involving partitions with " $n+1$ copies of n " and " $n+2$ copies of n ", respectively, are proved. These theorems have potential of yielding infinite Rogers-Ramanujan type identities on MacMahon's lines. Five particular cases are also discussed. Among them three are known and two provide new combinatorial interpretations of two known q-identities.

1. INTRODUCTION, DEFINITIONS, AND MAIN RESULTS

In this paper we shall prove some partition identities involving partitions with " $n+\ell$ copies on n ". We first recall the following definitions from [3]:

Definition 1. A partition with " $n+\ell$ copies of n ", $\ell \geq 0$ is a partition in which a part of size n, $n \geq 0$, can come in $n+\ell$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+\ell}$. In the part n_i, n can be zero if and only if $i \geq 1$. But in no partition are zeros permitted to repeat.

Definition 2. The weighted difference of two parts m_i, n_j, $m \geq n$ is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$.

Recently in [1] the following result was proved:

Theorem 1. *For $k \geq -3$, let $A_k(\nu)$ denote the number of partitions of ν with " n copies of n " such that the weighted difference of each pair of summands m_i, n_j is greater than k. Then*

$$\sum_{\nu=0}^{\infty} A_k(\nu) q^\nu = \sum_{\nu=0}^{\infty} \frac{q^{\nu\left[1+\frac{(k+3)(\nu-1)}{2}\right]}}{(q;q)_\nu (q;q^2)_\nu}, \tag{1.1}$$

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where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

Some particular cases were also discussed. Among them was the following identity in which the partitions enjoy a convolution property:

Corollary 1. *$A_{-2}(\nu)$ equals $\sum_{k=0}^{\nu} A_{\nu-k}B_k$, where A_ν denotes the number of partitions of ν into distinct parts $\equiv \pm 3 \pmod 7$ and B_ν denotes the number of partitions of ν into parts $\not\equiv 0, \pm 4 \pmod{14}$.*

Corollary 1 is a combinatorial interpretation of the identity [5, Eq. (3.1), p. 219]:

$$(1.2)\qquad \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n^2+n)}}{(q;q)_n(q;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1+q^n)(1-q^{7n-2})(1-q^{7n-5})(1-q^{7n})}{(1-q^n)(1+q^{7n-1})(1+q^{7n-6})}.$$

In this paper we shall prove the following two theorems, which are very similar to Theorem 1:

Theorem 2. *For $k \geq -3$, let $B_k(\nu)$ denote the number of partitions of ν with "$n+1$ copies of n" such that the weighted difference of each pair of parts is greater than k, the parts are nonnegative, and for some i, i_{i+1} is a part. Then*

$$(1.3)\qquad \sum_{\nu=0}^{\infty} B_k(\nu)q^\nu = \sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu+1)(k+3)/2}}{(q;q)_\nu(q;q^2)_{\nu+1}}.$$

Theorem 3. *For $k \geq -3$, let $C_k(\nu)$ denote the number of partitions of ν with "$n+2$ copies of n" into nonnegative parts such that the weighted difference of each pair of parts is greater than k, and for some i, i_{i+2} is a part. Then*

$$(1.4)\qquad \sum_{\nu=0}^{\infty} C_k(\nu)q^\nu = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]}}{(q;q)_\nu(q;q^2)_{\nu+1}}.$$

In §3 we shall show that Theorem 2 and 3 provide new combinatorial interpretations (similar to Corollary 1 above) of the following q-identities [6, p. 160 I(80) and I(82), respectively]:

$$(1.5)\qquad \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(q;q)_n(q;q^2)_{n+1}} = \frac{(-q;q)_\infty}{(q;q)_\infty} \prod_{n=1}^{\infty}(1-q^{7n-2})(1-q^{14n-11})(1-q^{14n-3})(1-q^{7n})$$

and

$$(1.6)\qquad \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+3)}}{(q;q)_n(q;q^2)_{n+1}} = \frac{(-q;q)_\infty}{(q;q)_\infty} \prod_{n=1}^{\infty}(1-q^{7n-3})(1-q^{7n-4})(1-q^{14n-13})(1-q^{14n-1})(1-q^{7n}).$$

In §3 we shall pose some very significant open problems.

2. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let $A_k(m,\nu)$ and $B_k(m,\nu)$ denote respectively the number of partitions of ν enumerated by $A_k(\nu)$ and $B_k(\nu)$ with the added restriction that there be exactly m parts.

It was shown in [1, Eq. (2.6)] that if

$$f_k(z,q)=\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}A_k(m,\nu)z^m q^\nu, \tag{2.1}$$

then

$$f_k(z,q)=\sum_{\nu=0}^{\infty}\frac{q^{\nu[1+(k+3)(\nu-1)/2]}z^\nu}{(q;q)_\nu(q;q^2)_\nu}; \tag{2.2}$$

and so

$$f_k(z,q)-f_k(zq,q)=zq\sum_{\nu=0}^{\infty}\frac{q^{\nu(\nu+1)(k+3)/2}(zq)^\nu}{(q;q)_\nu(q;q^2)_{\nu+1}}. \tag{2.3}$$

Setting

$$\begin{aligned}g_k(z,q)&=\sum_{\nu=0}^{\infty}\frac{q^{\nu(\nu+1)(k+3)/2}z^\nu}{(q;q)_\nu(q;q^2)_{\nu+1}}\\&=\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}E_k(m,\nu)z^m q^\nu,\end{aligned} \tag{2.4}$$

we see by coefficient comparision in (2.3) that

$$A_k(m,\nu)-A_k(m,\nu-m)=E_k(m-1,\nu-m). \tag{2.5}$$

Equation (2.5) shows that $E_k(m,\nu)$ equals the number of partitions of $\nu+m+1$ with "n copies n" into $m+1$ parts such that the weighted difference of each pair of parts m_i, n_j is greater than k and, for some i, i_i is a part. If we subtract 1 from each part of a partition enumerated by $E_k(m,\nu)$ ignoring the subscripts we see that the resulting partition is enumerated by $B_k(m+1,\nu)$. This implies that

$$E_k(m,\nu)=B_k(m+1,\nu). \tag{2.6}$$

Hence

$$\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}B_k(m+1,\nu)z^m q^\nu=\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{q^{\nu(\nu+1)(k+3)/2}z^m}{(q;q)_\nu(q;q^2)_{\nu+1}}. \tag{2.7}$$

Now

$$\sum_{\nu=0}^{\infty}B_k(\nu)q^\nu=\sum_{\nu=0}^{\infty}\left[\sum_{m=0}^{\infty}B_k(m,\nu)\right]q^n=g_k(1,q)=\sum_{\nu=0}^{\infty}\frac{q^{\nu(\nu+1)(k+3)/2}}{(q;q)_\nu(q;q^2)_{\nu+1}}.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Let $C_k(m, \nu)$ denote the number of partitions of ν counted by $C_k(\nu)$ with the added restriction that there be exactly m parts. Equation (2.3) can be written as

$$f_k(z, q) - f_k(zq, q) = zqh_k(z, q), \tag{2.8}$$

where

$$h_k(z, q) = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]} z^\nu}{(q;q)_\nu (q;q^2)_{\nu+1}}. \tag{2.9}$$

Setting

$$h_k(z, q) = \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} F_k(m, \nu) z^m q^\nu, \tag{2.10}$$

we see by coefficient comparison in (2.8) that

$$A_k(m, \nu) - A_k(m, \nu - m) = F_k(m-1, \nu-1). \tag{2.11}$$

Equation (2.11) shows that $F_k(m, \nu)$ equals the number of partitions of $\nu + 1$ with "n copies of n" into $m + 1$ parts such that the weighted difference of each pair of parts m_i, n_j is greater than k and for some i, i_i is a part. If we replace this part i_i by $(i-1)_{i+1}$, we see that the resulting partition is enumerated by $C_k(m+1, \nu)$. This implies that

$$F_k(m, \nu) = C_k(m+1, \nu). \tag{2.12}$$

Hence

$$\sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} C_k(m+1, \nu) z^m q^\nu = \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]} z^\nu}{(q;q)_\nu (q;q^2)_{\nu+1}}.$$

Now

$$\begin{aligned} \sum_{\nu=0}^{\infty} c_k(\nu) q^\nu &= \sum_{\nu=0}^{\infty} \left\{ \sum_{m=0}^{\infty} C_k(m, \nu) \right\} q^\nu = h_k(1, q) \\ &= \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]}}{(q;q)_\nu (q;q^2)_{\nu+1}}, \end{aligned}$$

and the proof of Theorem 3 is completed.

3. Particular cases

For $k = 0$, Theorem 2, in view of the identity [6, I(4), p. 156]

$$\sum_{n=0}^{\infty} \frac{q^{3n(n+1)/2}}{(q;q)_n (q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1 - q^{10n})(1 - q^{10n-2})(1 - q^{10n-8}), \tag{3.1}$$

yields

Corollary 2.1. *The number of partitions of ν with "$n+1$ copies of n" into nonnegative parts, such that each pair of parts m_i, n_j has positive weighted difference and for some i, i_{i+1} is a part, equals the number of ordinary partitions of ν into parts $\not\equiv 0$, $\pm 2 \pmod{10}$.*

For $k=-1$, Theorem 2, in view of the identity [6, I(60), p. 158]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n (q;q^2)_{n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{14n-4})(1-q^{14n-10})(1-q^{14n}), \tag{3.2}$$

reduces to

Corollary 2.2. *The number of partitions of ν with "$n+1$ copies of n" into nonnegative parts, such that each pair of parts m_i, n_j has nonnegative weighted difference and for some i, i_{i+1} is a part, equals the number of ordinary partitions of ν into parts $\not\equiv 0$, $\pm 4 \pmod{14}$.*

On the other hand, in this particular case, Theorem 3, in view of the identity [6, I(59), p. 157]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q;q)_n (q;q^2)_{n+1}} = \frac{1}{(q;q)_n} \prod_{n=1}^{\infty} (1-q^{14n-2})(1-q^{14n-12})(1-q^{14n}),$$

reduces to

Corollary 3.1. *The number of partitions of ν with "$n+2$ copies of n" into nonnegative parts, such that each pair of parts m_i, n_j has nonnegative weighted difference and for some i, i_{i+2} is a part, equals the number of ordinary partitions of ν into parts $\not\equiv 0$, $\pm 2 \pmod{14}$.*

Remark 1. Corollaries 2.1, 2.2, and 3.1 are the particular cases $k=2$, $\ell=1$; $k=3$, $\ell=1$; and $k=3$, $\ell=2$, respectively of Theorem 4 in [3].

Remark 2. Another proof of Corollaries 2.1, 2.2, and 3.1 can be found in [4]. For $k=-2$, Theorems 2 and 3 give the following combinatorial interpretations, presumably new, of the identities (1.5) and (1.6), respectively.

Corollary 2.3. *The number of partitions of ν with "$n+1$ copies of n" into nonnegative parts, such that the weighted difference of each pair of summands m_i, n_j is greater than or equal to -1 and for some i, i_{i+1} is a part, equals $\sum_{k=0}^{\nu} C_k D_{\nu-k}$, where C_k denotes the number of partitions k into parts $\equiv \pm 1$, ± 4, $\pm 6 \pmod{14}$ and D_k denotes the number of partitions of k into distinct parts.*

Example. $B_{-2}(5) = 12$, since the relevant partitions are 5_6, 5_1+0_1, 5_2+0_1, 5_3+0_1, 5_4+0_1, 5_5+0_1, 4_1+1_2, 4_2+1_2, $4_1+1_1+0_1$, $4_2+1_1+0_1$, $4_3+1_1+0_1$, $3_1+2_1+0_1$.

Also,

$$\sum_{k=0}^{5} C_k D_{5-k} = C_0D_5 + C_1D_4 + C_2D_3 + C_3D_2 + C_4D_1 + C_5D_0$$
$$= 1(3) + 1(2) + 1(2) + 1(1) + 2(1) + 2(1)$$
$$= 12.$$

Corollary 3.2. *The number of partitions of* ν *with "*$n+2$ *copies of* n*" into nonnegative parts, such that the weighted difference of each pair of summands* m_i, n_j *is greater than or equal to* -1 *and for some* i, i_{i+2} *is a part, equals* $\sum_{k=0}^{\nu} E_k D_{\nu-k}$, *where* E_k *denotes the number of partitions of* k *into parts* $\equiv 2,5,6,8,9,12 \pmod{14}$ *and* D_k, *as in Corollary* 2.3, *denotes the number of partitions of* k *into distinct parts.*

Example. $C_{-2}(6) = 10$, since the relevant partitions are 6_8, 6_1+0_2, 6_2+0_2, 6_3+0_2, 6_4+0_2, 6_5+0_2, $4_1+2_1+0_2$, $4_2+2_1+0_2$, 5_1+1_3, 5_2+1_3. Also,

$$\sum_{\nu=0}^{6} E_k D_{6-k} = E_0D_6 + E_1D_5 + E_2D_4 + E_3D_3 + E_4D_2 + E_5D_1 + E_6D_0$$
$$= 1(4) + 0(3) + 1(2) + 0(2) + 1(1) + 1(1) + 2(1)$$
$$= 10.$$

4. Conclusion

Many questions arise from this work. The most obvious among them are:

(1) Can Theorems 1, 2, and 3 be combined into one?

(2) Is it possible to give a nice combinatorial interpretation of Theorems 1, 2, and 3 for the general value of k?

(3) The methods used here were also used recently in [2] to give n-color partitions theoretic interpretations of several q-identities from [6]. Is it possible to prove Theorems 3 and 4 of [3] by using these methods?

(4) We have seen that each of Theorems 1, 2, and 3 yields a partition identity wherein the partitions enjoy a convolution property. Does this lead to the conclusion that there exists an infinite family of identities wherein partitions enjoy a convolution property?

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