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THE $T1$ THEOREM FOR MARTINGALES

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ABSTRACT. The $T1$ theorem of David and Journé gives necessary and sufficient conditions that a singular integral operator be bounded from $L^2(R^n)$ to $L^2(R^n)$. In this paper, the definition of singular integral operator is extended to the setting of operators on $L^2(\Omega)$ where Ω denotes Wiener space. The main theorem is that the $T1$ theorem holds in this new setting.

1. INTRODUCTION

The celebrated $T1$ theorem of David and Journé gives necessary and sufficient conditions for a singular integral operator to be bounded from $L^2(R^n)$ to $L^2(R^n)$. The definitions of the conditions involved in this theorem explicitly involve the dimension n. In this paper, we study the extension of the $T1$ theorem to an infinite dimensional setting—the Wiener space Ω, which is naturally isometric to R^∞ equipped with a Gaussian measure. With the appropriate definitions for this new setting, it will be shown that the $T1$ theorem holds for $L^2(\Omega)$ in addition to $L^2(R^n)$. This paper is a natural continuation of an earlier joint paper of R. Bañuelos and the author.

We start by stating the $T1$ theorem of David and Journé. We must first give several preliminary definitions.

Definition. An operator T is called a singular integral operator if T is a linear operator, which is continuous from the Schwartz class $\mathscr{S}(R^n)$ into $\mathscr{S}'(R^n)$ and there is a kernel $K(x,y)$ defined for $x \neq y$ in R^n and constants c and $0 < \delta \leq 1$ such that the following three properties hold:

(1) $|K(x,y)| \leq c|x-y|^{-n}$,
(2) for all $x_0, x, y \in R^n$, such that $|x_0 - x| < \frac{|x-y|}{2}$, $|K(x_0,y) - K(x,y)| + |K(y,x_0) - K(y,x)| \leq c|x_0 - x|^\delta |x-y|^{-(n-\delta)}$,
(3) for each pair f, ϕ, of disjointly supported $\mathscr{C}_0^\infty(R^n)$ functions, the evaluation of the distribution Tf on the test function ϕ is given by $Tf(\phi) = \iint K(x,y)f(y)\phi(x)\,dx\,dy$.

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Definition. We say a singular integral operator T has the weak boundedness property if for any bounded set $B \subset \mathscr{C}_0^\infty(R^n)$ there exists a constant c which depends only on B and T so that for all ϕ, $\psi \in B$, $x \in R^n$, and $t > 0$

$$|T\psi_t^x(\phi_t^x)| \le ct^{-n} \quad \text{where } \phi_t^z(x) = t^{-n}\phi\left(\frac{(z-x)}{t}\right).$$

Theorem (David and Journé). *A singular integral operator T can be extended to a bounded operator from $L^2(R^n)$ to $L^2(R^n)$ if and only if the following three conditions are satisfied*:

$T1 \in BMO$,
$T^*1 \in BMO$,
T has the weak boundedness property.

See [6] for a proof of this theorem. We will make several comments about the proof at this point. First, it should be noted that $T1$ is not initially defined since $1 \notin \mathscr{S}(R^n)$. It is possible to define $T1$, however, as a distribution on those test functions in $\mathscr{C}_0^\infty(R^n)$ with vanishing integral. The proof then proceeds by decomposing the operator T into three pieces, $T = A + Y + Z$. Y has the property that $Y1 = T1$ but $Y^*1 = 0$ while Z has the property that $Z1 = 0$ while $Z^*1 = T^*1$. These Y and Z can be defined using paraproducts. A is then just a singular integral operator with $A1 = 0$ and $A^*1 = 0$. Sufficiency of the three conditions is then established by considering each piece separately. Necessity follows from a standard computation. Since the dimension plays such an explicit role in the definition of singular integral operator, we will not try to extend this definition to the infinite dimensional case directly. Instead we shall take as the definition the decomposition used by David and Journé in their proof of the $T1$ theorem.

Let B_t be an n-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$, $\mathscr{F}_t = \sigma(B_s\colon 0 \le s \le t)$ and for any $X \in L^2(\Omega)$ we will use $t \mapsto X_t$ to denote the almost surely continuous version of $t \mapsto E[X|\mathscr{F}_t]$. Here we take Ω to be Wiener space, but this is not necessary. For any $X \in L^2(\Omega)$, there is an essentially unique $\mathscr{F}_t$-adapted R^n-valued map, $s \mapsto H_s$ such that $E[\int_0^\infty |H_s|^2\, ds] < \infty$ and $X_t = E[X] + \int_0^t H_s \cdot dB_s (\text{a.s.}P)$.

Definition. A process A is a *proper integrator* if $A\colon [0,\infty) \to R^n \otimes R^n$ is cadlag (i.e., right continuous with limits from the left), adapted to $\mathscr{F}_t$, and there is a $c > 0$ such that for every $X \in L^\infty(\Omega)$, $\|A * X\|_{L^2(\Omega)} \le c\|X\|_{L^\infty(\Omega)}$ where $A * X = \int_0^\infty A_t H_t \cdot dB_t$.

The operator $A * X$ is called the martingale transform of X by A. Martingale transforms were first introduced by Burkholder in [3].

Definition. A *singular integral operator* on $L^\infty(\Omega)$ is an operator of the form

$$T(X) = A * X + \int_0^\infty X_t\, dY_t + \langle X, Z\rangle_\infty, \tag{4}$$

where A is a proper integrator, $Y, Z \in L^2(\Omega)$ and $\langle X, Z\rangle_t$ is the covariance process of X and Z.

We will see later that the representation (4) of a singular integral operator is essentially unique. In this definition, we have used $A * X$, $\int_0^\infty X_t\, dY_t$, and $\langle X, Z\rangle_\infty$ to replace the operators A, Y, and Z respectively in the David-Journé decomposition of a singular integral operator. $\int_0^\infty X_t\, dY_t$ and $\langle X, Z\rangle_\infty$ are formed from probabilistic paraproducts as described in [2] and have the same properties as Y and Z in the David-Journé decomposition. $A*X$ has the same properties as the operator A in the David-Journé decomposition. Since we don't have a concept of smoothness for functions on Wiener space, we use $L^\infty(\Omega)$ as our set of test functions. This is reasonable since for a probability space Ω, $L^\infty(\Omega) \subset \bigcap_{1\le p<\infty} L^p(\Omega)$. The conditions on A, Y, and Z in this definition were chosen so that T will be bounded from $L^\infty(\Omega) \to L^2(\Omega)$. This definition of singular integral operator generalizes the stochastic versions of conjugate operators, Riesz transforms, operators of Laplace transform type, and paraproducts and remainder operators as considered in [1], [2], [5], [8] and [9].

Definition. A singular integral operator T is said to have the *weak boundedness property* if $\| \sup_{0\le t<\infty} \|A(t)\|_{\mathrm{op}}\|_\infty < \infty$, where $\| \ \|_{\mathrm{op}}$ denotes the operator norm on $R^n \otimes R^n$ and A is the proper integrator in the representation (4) of T.

Definition. $X \in BMO(\Omega)$ means $X \in L^2(\Omega)$ and there is a constant $c \ge 0$ such that for all stopping times τ, $E[|X - X_t||\mathscr{F}_t] \le c$ a.s. The square root of the smallest constant c for which this holds is the $BMO(\Omega)$ norm of X.

The definition of $BMO(\Omega)$ given here is standard (see [7] for example). We will also use the equivalent formulation $E[(X - X_\tau)^2] \le cP[\tau < \infty]$. The definition of weak boundedness is chosen to make the following theorem true.

Theorem 1. *A singular integral operator T can be extended to a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$ if and only if the following three conditions are satisfied*:

$T1 \in BMO$,
$T^*1 \in BMO$,
T has the weak boundedness property.

Theorem 1 and some corollaries will be proved in §2. In §3 we will examine contractions of singular integral operators.

2. The $T1$ Theorem for Martingales

Proof of Theorem 1. The sufficiency of the three conditions follows immediately from the results in [2] and the Burkholder-Gundy inequalities (see [4]). Since $T1 = Y$ and $T^*1 = Z$, the boundedness of the second and third parts of T is immediate from Theorem 1.1 of [2] and the first part of T is bounded by the Burkholder-Gundy inequalities. So we must now show the three conditions are necessary. We suppose $\|T(X)\|_2 \le k\|X\|_2$. We first show that T has the weak boundedness property. Suppose now that $\|\sup_{0\leq t<\infty}\|A(t)\|_{\text{op}}\| > k$. Then since A is cadlag, we can find a stochastic interval (σ, ρ) which is not a.s. empty such that for $\sigma < t < \rho$ we have $c < \|A(t)\|_{\text{op}} < 2c$ for some $c > k$. Define an R^n-valued function $v(t)$ by $|v(t)| = 1$ and $|A(t)v(t)| = \|A(t)\|_{\text{op}}$. Since $Y \in L^2(\Omega)$, we can find an adapted R^n-valued process K_t such that $Y_t = E[Y] + \int_0^t K_s \cdot dB_s$. Define

$$H_t = \begin{cases} |K_t|v(t), & t \in (\sigma, \rho) \\ 0, & t \notin (\sigma, \rho). \end{cases}$$

Now define $\overline{X} = \int_0^\infty H_t \cdot dB_t$. Since $\overline{X}_t$ and Z_t are continuous, for any ϵ we can find a stopping time τ with $\sigma < \tau \leq \rho$ such that for $\sigma < t < \tau$ we have $\|\overline{X}_t - \overline{X}_\sigma\| < \varepsilon$ a.s. and $\|Z_t - Z_\sigma\| < \varepsilon$ a.s. Finally, let $X = \overline{X}_\tau$. Observe that

$$\begin{aligned} \|X\|_2 &= \int_0^\tau |H_t|^2\, dt \\ &= \int_\sigma^\tau |H_t|^2\, dt \\ &= \int_\sigma^\tau |K_t|^2\, dt \\ &= E[(Y_\tau - Y_\sigma)^2]^{\frac{1}{2}}. \end{aligned}$$

In particular, note that $X \in L^2(\Omega)$. Now from $\|TX\|_2^2 \leq k^2\|X\|_2^2$ we obtain

$$\begin{aligned} (5) \qquad E[(A*X)^2] \leq k^2\|X\|_2^2 &+ 2\left|E\left[\left(\int_0^\infty A(t)H_t \cdot dB_t\right)\left(\int_0^\infty X_t\, dY_t\right)\right]\right| \\ &+ 2\left|E\left[\left(\int_0^\infty A(t)H_t \cdot dB_t\right)\langle X, Z\rangle_\infty\right]\right| \\ &+ 2\left|E\left[\left(\int_0^\infty X_t\, dY_t\right)\langle X, Z\rangle_\infty\right]\right|. \end{aligned}$$

We now proceed to estimate the various terms in (5).

$$\begin{aligned} (6) \qquad E[(A*X)^2] &= E\left[\left(\int_0^\infty A(t)H_t \cdot dB_t\right)^2\right] \\ &= E\left[\int_0^\infty |A(t)H_t|^2\, dt\right] \\ &\geq c^2 E\left[\int_0^\infty |H_t|^2\, dt\right] \\ &= c^2\|X\|_2^2, \end{aligned}$$

(7)
$$\begin{aligned}\left|E\left[(A*X)\left(\int_0^\infty X_t dY_t\right)\right]\right| &= \left|E\left[\left(\int_0^\infty A(t)H_t\cdot dB_t\right)\left(\int_0^\infty X_t dY_t\right)\right]\right|\\ &= \left|E\left[\left(\int_0^\infty A(t)H_t\cdot dB_t\right)\left(\int_0^\infty (X_tK_t)\cdot dB_t\right)\right]\right|\\ &= \left|E\left[\int_0^\infty (A(t)H_t)\cdot(X_tK_t)\,dt\right]\right|\\ &\le E\left[\int_0^\infty (\|A(t)\|_{op}|X_t|)|H_t\cdot K_t|\,dt\right]\\ &\le 2c\varepsilon E\left[\int_\sigma^\tau |H_t|^2\,dt\right]\\ &= 2c\varepsilon\|X\|_2^2,\end{aligned}$$

(8)
$$\begin{aligned}|E[(A*X)\langle X,Z\rangle_\infty^2]| &= \left|E\left[\left(\int_0^\infty A(t)H_t\cdot dB_t\right)\langle X,Z\rangle_\infty^2\right]\right|\\ &\le \left\|\int_0^\infty A(t)H_t\cdot dB_t\right\|_2 \|\langle X,Z\rangle_\infty\|_2\\ &\le (2c\|X\|_2)(\varepsilon\|X\|_2)\\ &= 2c\varepsilon\|X\|_2^2,\end{aligned}$$

(9)
$$\begin{aligned}2\left|E\left[\left(\int_0^\infty X_t dY_t\right)\langle X,Z\rangle_\infty\right]\right| &= \left|E\left[Z\left(\int_0^\infty\left(\int_0^t X_s dY_s\right)dX_t\right)\right]\right|\\ &= \left|E\left[Z\left(\int_\sigma^\tau\left(\int_\sigma^{t\wedge\tau} X_s dY_s\right)dX_t\right)\right]\right|\\ &= \left|E\left[Z\left(\int_0^\infty\left(\int_\sigma^{t\wedge\tau} X_s dY_s\right)dX_t\right)\right]\right|\\ &= \left|E\left[\left(\int_\sigma^\tau X_t dY_t\right)\langle X,Z\rangle_\infty\right]\right|\\ &\le \left\|\int_\sigma^\tau X_s dY_s\right\|_2 \|\langle X,Z\rangle_\infty\|_2\\ &\le \varepsilon\|Y_\tau - Y_\sigma\|_2\varepsilon\|X\|_2\\ &= \varepsilon^2\|X\|_2^2,\end{aligned}$$

where the first and fourth equalities follow from Theorem 2.1(ii) of [2].

Now using estimates (6), (7), (8), and (9) in (5) we obtain $c^2\|X\|_2^2 \le [(k^2 + \varepsilon^2) + 8c\varepsilon]\|X\|_2^2$. Since $c > k$ and ε is arbitrary, this implies that $\|X\|_2 = 0$ and therefore Y_t must be constant on the stochastic interval (σ, τ). We now repeat the construction of $\overline{X}$ using Z instead of Y. Then we can find a stopping time τ' such that $\sigma < \tau'$ a.s. and for $\sigma \le t < \tau'$, $|\overline{X}_t - \overline{X}_\sigma| < \varepsilon$ a.s. and $|Z_t - Z_\sigma| < \varepsilon$ a.s. Finally, we define a new $X = \overline{X}_{\tau\wedge\tau'}$. In this case, the inequality $\|TX\|_2 \le k\|X\|_2$ yields

$$E[(A * X)^2] \le k^2\|X\|_2^2 + 2|E[(A * X)\langle X, Z\rangle_\infty]| \tag{10}$$

in the same manner as we obtained (5). Then from (6) and (8) we obtain $c^2\|X\|_2^2 \le (k^2+\varepsilon^2)\|X\|_2^2$ and again since $c > k$ and ε is arbitrary, this implies that $\|X\|_2 = 0$ and hence that Z_t is also constant on the stochastic interval $(\sigma, \tau \wedge \tau')$. Finally, we define $X = \int_\sigma^{\tau\wedge\tau'} v(t)\cdot dB_t$ and then $TX = A * X$, and a simple computation gives $\|A * X\|_2 \ge c\|X\|_2 > k\|X\|_2$. But this contradicts the assumption that T is bounded with constant k. It follows that $\|\sup_{0\le t<\infty} \|A(t)\|_{\mathrm{op}}\|_\infty \le k$, and so T must be weakly bounded. This implies that $\|A * X\|_2 \le k\|X\|_2$ and we can now use this inequality in conjunction with $\|TX\|_2 \le k\|X\|_2$ to derive

$$\left\|\int_0^\infty X_t\,dY_t + \langle X, Z\rangle_\infty\right\|_2 \le 2k\|X\|_2. \tag{11}$$

We now show that this implies $Y = T1 \in BMO$. Let τ be an arbitrary stopping time and let $\rho = \inf\{t > \tau : |Y_t - Y_\tau| \ge \varepsilon\}$. Then since Y_t is continuous, we have $|Y_\rho - Y_\tau| \le \varepsilon$ and $|Y_\rho - Y_\tau| = \varepsilon$ if $\rho < \infty$. Also $|Y_\rho - Y_\tau| = 0$ if $\tau = \infty$. Now let $X = Y_\rho - Y_\tau$ so $X_t = Y_{t\wedge\rho} - Y_{t\wedge\tau}$. Then from (11) we obtain

$$E\left[\left(\int_0^\infty X_t\,dY_t\right)^2\right] \le 4k^2\|X\|_2^2 + 2\left|E\left[\left(\int_0^\infty X_t\,dY_t\right)\langle X, Z\rangle_\infty\right]\right|. \tag{12}$$

Now

$$\begin{aligned} E\left[\left(\int_0^\infty X_t\,dY_t\right)^2\right] &= E\left[\int_\tau^\infty X_t^2\,d\langle Y\rangle_t\right] \\ &= E\left[\int_\tau^\rho X_t^2\,d\langle Y\rangle_t + \int_\rho^\infty \varepsilon^2\,d\langle Y\rangle_t\right] \\ &\ge \varepsilon^2 E\left[\int_\rho^\infty d\langle Y\rangle_t\right] \\ &= \varepsilon^2 E[(Y - Y_\rho)^2], \end{aligned} \tag{13}$$

$$E[X^2] \le \varepsilon^2 P[\tau < \infty], \tag{14}$$

$$\begin{aligned} \left|E\left[\left(\int_0^\infty X_t\,dY_t\right)\langle X, Z\rangle_\infty\right]\right| &= \left|E\left[Z\int_0^\infty \left(\int_0^t X_s\,dY_s\right)dX_t\right]\right| \\ &= \left|E\left[Z\int_\tau^\rho \left(\int_\tau^t X_s\,dX_s\right)dX_t\right]\right| \\ &\le \|Z\|_2 \left\|\int_\tau^\rho \left(\int_\tau^t X_s\,dX_s\right)dX_t\right\|_2. \end{aligned} \tag{15}$$

But $\|\int_0^\infty U\,dV\|_2 \le \|U\|_4\|V\|_4$ and $\|\int_0^\infty U\,dV\|_4 \le \|U\|_8\|V\|_8$ from [2] and $|X| \le \varepsilon$ implies $\|X\|_4$, $\|X\|_8 \le \varepsilon$. Plugging these facts into equation (15) yields

$$\left|E\left[\left(\int_0^\infty X_t\,dY_t\right)\langle X, Z\rangle_\infty\right]\right| \le c\|Z\|_2\varepsilon^3. \tag{16}$$

Now substituting (13), (14), and (16) into (12) gives

$$E[(Y - Y_\rho)^2] \leq 4k^2 P[\tau < \infty] + 2c\varepsilon \|Z\|_2 . \tag{17}$$

Now if we let $\varepsilon \to 0$ then $\rho \to \tau$ and (17) becomes

$$E[(Y - Y_\tau)^2] \leq 4k^2 P[\tau < \infty] . \tag{18}$$

So $Y = T1 \in BMO$ with $\|T1\|_{BMO} \leq 2k$. Finally, we now have $\|\langle X , Z\rangle\|_2 \leq 4k\|X\|_2$, but Theorem 2.3 of [2] then asserts that $Z \in BMO$ (with $\|Z\|_{BMO} \leq (4k)^2 + 1$). Since $T^* 1 = Z$, this completes the proof of Theorem 1.

Corollary 1. *The representation* (4) *of a singular integral operator is essentially unique.*

Proof of Corollary 1. If

$$\begin{aligned} TX &= A * X + \int_0^\infty X_t \, dY_t + \langle X , Z\rangle_\infty \\ &= \overline{A} * X + \int_0^\infty X_t \, d\overline{Y}_t + \langle X , \overline{Z}\rangle_\infty \end{aligned}$$

then

$$(A - \overline{A}) * X + \int_0^\infty X_t \, d(Y - \overline{Y})_t + \langle X , (Z - \overline{Z})\rangle_\infty \equiv 0 ,$$

so $\| \sup_{0 \leq t < \infty} \|A - \overline{A}\|_{\mathrm{op}} \|_\infty = 0$ and $\|Y - \overline{Y}\|_{BMO} = 0$ directly from the proof of the theorem and hence $A(t) = \overline{A}(t)$ for all $0 \leq t < \infty$ a.s. and $Y = \overline{Y}$ a.s. Then $\langle X , Z - \overline{Z}\rangle_\infty = 0$ for all $X \in L^2(\Omega)$ so $Z = \overline{Z}$ a.s.

Corollary 2. *If a singular integral operator T is bounded from $L^2(\Omega)$ to $L^2(\Omega)$, then it is bounded from $L^p(\Omega)$ to $L^p(\Omega)$ for all $1 < p < \infty$.*

Proof of Corollary 2. This follows immediately from the Burkholder-Gundy inequalities and Theorem 1.1 of [2].

Corollary 2 asserts that the usual Calderón-Zygmund theory of singular integral operators remains valid in the infinite dimensional case.

3. Contractions of singular integral operators

Much of the interest in the martingale transforms and paraproducts that make up the building blocks of singular integral operators lies in the fact that one can obtain classically interesting operators from them.

Definition. Let $\mathscr{T} \subset \mathscr{F}_\infty = \bigvee_{t=0}^\infty \mathscr{F}_t$. If T is a singular integral operator then we define the *contraction* of T with respect to $\mathscr{T}$ by $T_{\mathscr{T}} \colon L^\infty(\mathscr{T}) \to L^2(\mathscr{T})$ where $T_{\mathscr{T}} \colon X \mapsto E[TX|\mathscr{T}]$ where $L^p(\mathscr{T})$ denotes the set of all functions in $L^p(\Omega)$ that are $\mathscr{T}$ measurable.

Theorem 2. *If a singular integral operator is bounded $L^2(\Omega)$ to $L^2(\Omega)$ then its contractions are all bounded $L^p(\mathscr{T})$ to $L^p(\mathscr{T})$ for all $1 < p < \infty$.*

Proof of Theorem 2. This follows immediately from Corollary 2 and the fact that conditional expectation is a contraction on L^p.

Example 1. Let D be a domain with smooth boundary in R^n and let $x_0 \in D$. Let μ denote harmonic measure on ∂D with respect to the point x_0. Let B_t be a Brownian motion starting at x_0 and let τ be the first exit time of B_t from D. Let $\mathscr{T} = \sigma(B_\tau)$ where $\sigma(B_\tau)$ is the smallest σ-field such that B_τ is measurable with respect to $\sigma(B_\tau)$. Then the contraction of a singular integral operator with respect to $\mathscr{T}$ is an operator which can be defined from $L^\infty(\partial D, \mu)$ to $L^2(\partial D, \mu)$. This follows from the fact that the mapping $f \mapsto f(B_\tau)$ is an isometric isomorphism from $L^p(\partial D, \mu)$ to $L^p(\mathscr{T})$.

The importance of Theorem 2 and Example 1 is that many classically interesting operators can be obtained as contractions of singular integral operators on $L^2(\Omega)$ and then Theorem 2 can be used to prove the boundedness of these operators. In addition, given a family of singular integral operators defined for different R^n (e.g., the Riesz transforms) which can be defined as contractions of singular integral operators on $L^2(\Omega)$ with uniform bound, Theorem 2 then proves boundedness with constants independent of the dimension. Special cases of this technique have been used frequently in the past, see [1], [2], [5], [8], and [9]. Unfortunately, contractions do not preserve the necessity of the conditions in Theorem 1. We now give an example of an unbounded singular integral operator T and a σ-field $\mathscr{T}$ such that the contraction of T to $L^\infty(\mathscr{T})$ extends to a bounded operator from $L^2(\mathscr{T})$ to $L^2(\mathscr{T})$.

Example 2. Let D be the unit disk in R^2. Let B_t be a Brownian motion starting at the origin and let τ be the first exist time of B_t from D. Let

$$A(t) = \begin{cases} (\frac{1}{2} - |B_t|)^{-\frac{1}{4}}, & \text{when } |B_t| < \frac{1}{2} \text{ and } t \le \tau \\ 0, & \text{when } |B_t| \ge \frac{1}{2} \text{ or } t > \tau. \end{cases}$$

Let the singular integral operator T be defined by $T = A * X$. Suppose $f, g \in \mathscr{C}_0^\infty(\Sigma)$ where Σ is the unit circle and let X and Y be the random variables $f(B_\tau)$ and $g(B_\tau)$ respectively. Let $\mathscr{T} = \sigma(B_\tau)$. Let u, v be the Poisson integrals of f, g respectively. Note that if

$$f = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) \tag{19}$$

then

(20)
$$u = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

We now compute

$$\begin{aligned}
(T_{\mathcal{J}}X, Y) &= \int_0^{2\pi} E[Tf|B_\tau = x]g(x)\,dx \\
&= \int_0^{2\pi} E[g(B_\tau)Tf|B_\tau = x]\,dx \\
&= E[g(B_\tau)Tf] \\
&= E\left[\left(\int_0^\tau \nabla v(B_t)\cdot dB_t\right)\left(\int_0^\tau (A(t)\nabla u(B_t))\cdot dB_t\right)\right] \\
&= E\left[\int_0^\tau (A(t)\nabla u(B_t))\cdot \nabla v(B_t)\,dt\right] \\
&= \int_0^{\frac{1}{2}}\int_0^{2\pi} \left(\frac{1}{2}-r\right)^{-\frac{1}{4}} \ln(r)\nabla u(r,\theta)\cdot\nabla v(r,\theta) r\,d\theta\,dr \\
&= \int_0^{\frac{1}{2}}\int_0^{2\pi} \left(\frac{1}{2}-r\right)^{-\frac{1}{4}} \ln(r)\left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\frac{\partial v}{\partial \theta}\right) r\,d\theta\,dr \\
&= \int_0^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left(\frac{1}{2}-r\right)^{-\frac{1}{4}} \ln(r)2\pi n r^{2n-1}(a_n\alpha_n + b_n\beta_n)\right)dr \\
&= \sum_{n=1}^{\infty}\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-r\right)^{-\frac{1}{4}} \ln(r)2\pi n r^{2n-1}\,dr\right)(a_n\alpha_n + b_n\beta_n)\right\} \\
&= \sum_{n=1}^{\infty}((m_n a_n)\alpha_n + (m_n b_n)\beta_n),
\end{aligned}$$

where $m_n = \int_0^{1/2}(\frac{1}{2}-r)^{-1/4}\ln(r)2\pi n r^{2n-1}\,dr$ and a_n, b_n are the Fourier coefficients of f, and α_n, β_n are the Fourier coefficients of g. Then by the Plancherel theorem, it follows that $T_{\mathcal{J}}$ is a multiplier operator with multiplier m_n. To show that $T_{\mathcal{J}}$ is bounded it then suffices to show that m_n is bounded. To see this, we write

(21)
$$m_n = \int_0^{\frac{1}{4}}\left(\frac{1}{2}-r\right)^{-\frac{1}{4}}\ln(r)2\pi n r^{2n-1}\,dr + \int_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{1}{2}-r\right)^{-\frac{1}{4}}\ln(r)2\pi n r^{2n-1}\,dr.$$

Then we approximate the two integrals in (21) as follows.

$$\begin{aligned}\left|\int_0^{\frac14}\left(\frac12-r\right)^{-\frac14}\ln(r)2\pi n r^{2n-1}\,dr\right| &\le \int_0^{\frac14}\left|\left(\frac14\right)^{-\frac14}\ln(r)2\pi n r^{2n-1}\right|dr\\ &= 2\pi n\left(\frac14\right)^{-\frac14}\int_0^{\frac14}|r^{2n-1}\ln(r)|\,dr\\ &\le 2\pi n\left(\frac14\right)^{-\frac14}\int_0^{\frac14}r^{2n-2}\,dr\\ &= 2\pi n\left(\frac14\right)^{-\frac14}(2n-1)^{-1}\left(\frac14\right)^{2n-1}\\ &\le 2\pi\left(\frac14\right)^{-\frac14},\end{aligned}$$

$$\begin{aligned}\left|\int_{\frac14}^{\frac12}\left(\frac12-r\right)^{-\frac14}\ln(r)2\pi n r^{2n-1}\,dr\right| &\le \int_{\frac14}^{\frac12}\left|\left(\frac12-r\right)^{-\frac14}\ln\left(\frac14\right)2\pi n\left(\frac12\right)^{2n-1}\right|dr\\ &= \frac43\left(\frac14\right)^{\frac34}\left|\ln\left(\frac14\right)\right|2\pi n\left(\frac12\right)^{2n-1}\\ &\le \frac43\left(\frac14\right)^{\frac34}\left|\ln\left(\frac14\right)\right|2\pi.\end{aligned}$$

So $|m_n| \le 2\pi(\frac14)^{-\frac14} + \frac43(\frac14)^{\frac34}|\ln(\frac14)|2\pi$ for all n and we are done.

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