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REMARKS ON RINGS OF CONSTANTS OF DERIVATIONS

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ABSTRACT. Let k be a field of characteristic $p > 0$ and $D \neq 0$ a family of k-derivations of $k[x,y]$. We prove that $k[x,y]^D$,the ring of constants with respect to D, is a free $k[x^p,y^p]$-module of rank p or 1 and $k[x,y]^D = k[x^p,y^p,f_1,\dots,f_{p-1}]$ for some $f_1,\dots,f_{p-1} \in k[x,y]^D$.

In this note we generalize some results of A. Nowicki and M. Nagata on rings of constants for k-derivations of $k[x,y]$ in their paper [N-N]. Before stating their results let us set up notations which will be used throughout this paper.

We assume that k is always a field of characteristic $p > 0$. A k-linear map $d: k[x_1,\dots,x_n] \to k[x_1,\dots,x_n]$ is called a k-derivation of $k[x_1,\dots,x_n]$ if d satisfies the condition:

$$d(fg) = d(f)g + fd(g), \qquad \forall f,\ g \in k[x_1,\dots,x_n].$$

If D is a family of k-derivations of $k[x_1,\dots,x_n]$, we denote by $k[x_1,\dots,x_n]^D$ the set of constants of D in $k[x_1,\dots,x_n]$, i.e.,

$$k[x_1,\dots,x_n]^D = \{f \in k[x_1,\dots,x_n] \mid d(f) = 0,\ \forall d \in D\}.$$

If $D = \{d\}$, we simply write $k[x_1,\dots,x_n]^D = k[x_1,\dots,x_n]^d$.

It is easy to see that for any family D of k-derivations of $k[x_1,\dots,x_n]$, $k[x_1^p,\dots,x_n^p] \subseteq k[x_1,\dots,x_n]^D$; and $k[x_1,\dots,x_n]^D$, as a $k[x_1^p,\dots,x_n^p]$-submodule of $k[x_1,\dots,x_n]$, is a finitely generated $k[x_1^p,\dots,x_n^p]$-module. It is interesting to consider relations between these two rings. Among other things, in their recent paper [N-N] Nowicki and Nagata prove the following results:

A. [N-N, Proposition 4.2] *If $D \neq 0$, $p = 2$, then there exists an $f \in k[x,y]$ such that $k[x,y]^D = k[x^p,y^p,f]$;*

B. [N-N, Theorem 4.4] $k[x,y]^d$ *is a free $k[x^p,y^p]$-module for any single k-derivation d.*

In this note we generalize these two results; namely, we prove the following theorem.

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Theorem. *Let k be a field of characteristic $p > 0$, $D \neq 0$ a family of k-derivations of $k[x, y]$. Then*

(i) *$k[x, y]^D$ is a free $k[x^p, y^p]$-module of rank p or 1;*
(ii) *there exist $g_1, \ldots, g_{p-1} \in k[x, y]^D$ such that $k[x, y]^D = k[x^p, y^p, g_1, \ldots, g_{p-1}]$.*

Note that $k[x, y]^D$ is always a two-dimensional normal domain; hence it is Cohen-Macaulay. It is a natural question whether $k[x, y]^D$ is regular. We shall give a negative answer to this question by giving an example which is not locally UFD. We shall also give an example showing that for $n \geq 3$, $k[x_1, \ldots, x_n]^D$ may not be Cohen-Macaulay.

To prove the theorem, we need the following theorem (see [M], p. 140).

Theorem A. *Let A be a regular local ring, and B a domain containing A which is a finite A-module. Then B is free over A iff B is Cohen-Macaulay.*

Proof of theorem. Denote $A = k[x, y]^D$. A is a finitely generated $k[x^p, y^p]$-module, as being a $k[x^p, y^p]$-submodule of $k[x, y]$. Let $\mathscr{M}$ be a maximal ideal of $k[x^p, y^p]$. Then $k[x^p, y^p]_{\mathscr{M}}$ is a regular local ring, $A_{\mathscr{M}}$ is a finite $k[x^p, y^p]_{\mathscr{M}}$-module. Since A is a normal domain, $A_{\mathscr{M}}$ is normal. Note that $\dim A_{\mathscr{M}} = 2$. Hence, by [M, Theorem 38, p. 124], $A_{\mathscr{M}}$ is Cohen-Macaulay. Therefore, by using Theorem A, $A_{\mathscr{M}}$ is a free $k[x^p, y^p]_{\mathscr{M}}$-module. It follows that A is projective over $k[x^p, y^p]$. Thus, by Seshadri [S] (see [L]), $A = k[x, y]^D$ is a free $k[x^p, y^p]$-module.

Assume that $k[x, y]^D$, as a free $k[x^p, y^p]$-module, has rank s. Let $f_1, \ldots, f_s \in k[x, y]^D$ be a set of generators of $k[x^p, y^p]$-module $k[x, y]^D$. Denote $F = k(x^p, y^p)[f_1, \ldots, f_s]$. Then F is the field of quotients of $k[x, y]^D$ since $f_1, \ldots, f_s$ are integral over $k(x^p, y^p)$. Denote $t = [F : k(x^p, y^p)]$. Then $t = 1$, p, or p^2. But $t = p^2$ is impossible, as $D \neq 0$. Note that

$$F = k(x^p, y^p) \otimes_{k[x^p, y^p]} k[x, y]^D$$

and that $k[x, y]^D$ is a free $k[x^p, y^p]$-module. If follows that $f_1, \ldots, f_s$ is a basis of F over $k(x^p, y^p)$, and hence $s = t$. Therefore $s = 1$ or p, proving (i).

To prove (ii), we may assume that $s = p$ and $k[x, y]^D$ is generated by $f_1, \ldots, f_p$ as a $k[x^p, y^p]$-module. We have

$$1 = \sum_{i=1}^{p} r_i f_i, \quad r_i \in k[x^p, y^p], \qquad i = 1, \ldots, p.$$

This means that $(r_1, \ldots, r_p)$ is a unimodular row over $k[x, y]$. Since $k[x, y]$ is integral over $k[x^p, y^p]$, $(r_1, \ldots, r_p)$ must be a unimodular row over $k[x^p, y^p]$. Using Seshadri [S] (see [L]) again, $(r_1, \ldots, r_p)$ can be completed as a $p \times p$ matrix over $k[x^p, y^p]$ with determinant in k^*. Then a basis change gives a generating set $\{g_1 = 1, g_2, \dots, g_p\}$ of $k[x,y]^D$ as a $k[x^p, y^p]$-module. Hence $k[x,y]^D = k[x^p, y^p, g_2, \dots, g_p]$. □

Note that for any family D of k-derivations, $k[x,y]^D$ is a two-dimensional normal domain; hence it is Cohen-Macaulay. We here give an example showing that $k[x,y]^D$ may not be regular. Note that if k is of characteristic zero, $k[x,y]^D$ is always regular (see [N-N]).

Example 1. [N-N, Example 4.3] Let k be a field of characteristic $p > 0$ and let d be the k-derivation of $k[x,y]$ defined by

$$d(f) = \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y, \qquad f \in k[x,y].$$

We are going to show that $k[x,y]^d$ is not locally UFD. This implies that $k[x,y]^d$ is not regular. Let $\mathscr{M} = (x,y)k[x,y] \cap k[x,y]^d$. Since $k[x,y]$ is integral over $k[x,y]^d$, $\mathscr{M}$ is a maximal ideal of $k[x,y]^d$. Denote $A = k[x,y]^d$. We claim that $A_{\mathscr{M}}$ is not a UFD. It is easy to check that $x^p, y^p, x^{p-1}y$, $xy^{p-1} \in A$. They are all irreducible elements in A since any proper factor of them is not in $k[x,y]^d = A$. We want to prove that $x^p, y^p, x^{p-1}y$, and xy^{p-1} are all irreducible in $A_{\mathscr{M}}$. Suppose, for example, that $x^{p-1}y$ is reducible in $A_{\mathscr{M}}$. Then

$$x^{p-1}y = \frac{g_1}{h_1} \cdot \frac{g_2}{h_2}$$

where $g_1, g_2 \in \mathscr{M}$, $h_1, h_2 \in A - \mathscr{M}$. Hence

$$g_1 g_2 = x^{p-1} y h_1 h_2.$$

We may assume that $y | g_2$ (in $k[x,y]$). Then $g_1 = x^r h$, where $1 \le r \le p-1$ and $h \in k[x,y]$ such that $h | h_1 h_2$ (in $k[x,y]$). Since $h_1, h_2 \in A - \mathscr{M}$, $h_1 h_2 \notin (x,y)k[x,y]$, hence $h \notin (x,y)k[x,y]$. We have $d(x^r h) = d(g_1) = 0$ since $g_1 \in A$. But

$$\begin{aligned} d(x^r h) &= \frac{\partial}{\partial x}(x^r h)x + \frac{\partial}{\partial y}(x^r h)y \\ &= r x^r h + x^{r+1}\frac{\partial h}{\partial x} + x^r y \frac{\partial h}{\partial y}. \end{aligned}$$

Therefore

$$rh = -x\frac{\partial h}{\partial x} - y\frac{\partial h}{\partial y} \in (x,y)k[x,y].$$

Noticing that $r \not\equiv 0(p)$, we have $h \in (x,y)k[x,y]$, which is a contradiction. Hence $x^{p-1}y$ is irreducible in $A_{\mathscr{M}}$. Similarly, x^p, y^p, and xy^{p-1} are all irreducible in $A_{\mathscr{M}}$. We have

$$(xy)^p = x^p \cdot y^p = (x^{p-1}y) \cdot (xy^{p-1}).$$

To prove that $A_{\mathscr{M}}$ is not UFD, it suffices to prove that neither $x^p / x^{p-1}y$ nor x^p / xy^{p-1} is a unit in $A_{\mathscr{M}}$. But $x^p/x^{p-1}y = x/y$, $x^p/xy^{p-1} = x^{p-1}/y^{p-1}$.

Therefore, to prove that $A_{\mathscr{M}}$ is not UFD, it suffices to prove that $x^{p-1}/y^{p-1} \notin A_{\mathscr{M}}$. Suppose $x^{p-1}/y^{p-1} \in A_{\mathscr{M}}$. Then there exists an $f \in k[x, y]$ such that $x^{p-1}f \in A$, $y^{p-1}f \in A - \mathscr{M}$. But $y^{p-1}f \in (x, y)k[x, y]$, implying that $y^{p-1}f \in (x, y)k[x, y] \cap A = \mathscr{M}$. This is a contradiction, proving the claim that $A_{\mathscr{M}}$ is not UFD. □

If $n \geq 3$, $k[x_1, \ldots, x_n]^D$ may not be Cohen-Macaulay, as will be shown by the following example:

Example 2 [N-N, Example 4.6]. Let $n \geq 3$, and let d be the k-derivation of $k[x_1, \ldots, x_n]$ defined by

$$d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i^p, \qquad f \in k[x_1, \ldots, x_n].$$

Denote $A = k[x_1^p, \ldots, x_n^p]$, $B = k[x_1, \ldots, x_n]^d$. It is proved in [N-N] that B is not a free A-module. There exists a maximal ideal $\mathscr{M}$ of A such that $B_{\mathscr{M}}$ is not a free $A_{\mathscr{M}}$-module. Using Theorem A, $B_{\mathscr{M}}$ is not Cohen-Macaulay. Hence there exists a maximal ideal $\mathscr{N} \subset B$ such that $\mathscr{N} \cap (A - \mathscr{M}) = \varnothing$, $(B_{\mathscr{M}})_{\mathscr{N}} = B_{\mathscr{N}}$ is not Cohen-Macaulay. Therefore B is not Cohen-Macaulay. □

Indeed one can prove, using Theorem A, that for any D, $k[x_1, \ldots, x_n]^D$ is Cohen-Macaulay if and only if it is free over $k[x_1^p, \ldots, x_n^p]$.

References

[L] T. Y. Lam, *Serre's conjecture*, Lecture Notes in Math., no. 635, Springer-Verlag, New York, 1978.

[M] H. Matsumura, *Commutative algebra*, 2nd edition, Benjamin Cummings, 1980.

[N-N] A. Nowicki and M. Nagata, *Rings of constants for k-derivations in* $k[x_1, \ldots, x_n]$, J. Math. Kyoto Univ. **28** (1988), 111–118.

[S] C. S. Seshadri, *Triviality of vector bundles over the affine space* K^2, Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 456–458.

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