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# A NEW PROOF OF UNIQUENESS FOR MULTIPLE TRIGONOMETRIC SERIES

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ABSTRACT. Georg Cantor's 1870 theorem that an everywhere convergent to zero trigonometric series has all its coefficients equal to zero is given a new proof. The new proof uses the first formal integral of the series, while Cantor's proof used the second formal integral.

In 1870 Georg Cantor proved the following uniqueness theorem:

**Theorem** (Cantor [3]). *If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers and if*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{1}$$

*converges at each $x$ to $0$, then the series vanishes identically*; *i.e., all its coefficients are* $0$.

Cantor's proof used an idea of Riemann: That much of the behavior of (1) can be inferred from studying its formal second integral, $\frac{1}{4}a_0x^2 - \sum_{n=1}^{\infty}(a_n \cos nx + b_n, \sin nx)/n^2$. For this proof see [7, p. 326], [1, pp. 1–4] or [3]. The idea of the present work is to use the *first* formal integral, $L(x) := \frac{1}{2}a_0x + \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx)/n$. ($L$ is for Lebesgue.) Zygmund points out that the difficulty in using $L(x)$ is that $L(x)$ need not converge everywhere even if the series (1) does. (For example $\sum \sin nx/\log n$ converges everywhere but $-\sum \cos n0/n \log n$ diverges.) Nevertheless, here is a proof, dedicated to the would be extenders of Cantor's theorem, which uses $L$.

*Proof*. By the theorem of Cantor-Lebesgue ([7, p. 316], [1, Appendix 1] or [2]), $a_n$ and $b_n$ tend to 0 as $n$ tends to $\infty$ so that the coefficients of the series part of $L$ are $o(1/n)$, whence the sum of their squares is finite. By the Riesz-Fisher Theorem, this series represents an $L^2$ function. ([7, p. 127]) A theorem of Rajchman and Zygmund says that at every point the symmetric approximate derivative of $L$ is equal to (the value of (1) which is) $0$. ([7, p. 324]) We may also assume that $L$ is approximately continuous at every

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point of a $2\pi$-periodic set $E$ of full Lebesgue measure, since every measurable function is approximately continuous a.e. ([6, Vol. 2, p. 257]). Since $L(x)$ is approximately continuous and has non-negative symmetric derivative on $E$, by a recent elementary but ingenious and difficult result of C. Freiling and D. Rinne, $L$ is non-decreasing on $E$ ([4] and [5, Theorem 2]). Symmetrically $L$ is non-increasing on $E$, so that there is a constant $c$ with $L(x) = c$ for all $x$ in $E$. In other words, for all $x \in E$,

$$-c + \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx)/n = -\frac{1}{2}a_0 x. \tag{2}$$

The left side of equation (2) represents an $L^2$ function and is therefore Abel summable a.e. to that function ([7, p. 90, Equation 3.9 and p. 80, Equation 1.33]). At each point of Abel summability Tauber's original Tauberian theorem (recall the coefficients are $o(1/n)$) guarantees convergence ([7, p. 81]). Fix one such point $x_0$ which is also in $E$. Since equation (2) holds at $x_0$ and at $x_0+2\pi$ and the left side has the same value at both points, it follows that $a_0 = 0$. This means that the $L^2$ function represented by the left side of equation (2) is $0$ a.e. Bessel's inequality ([7, p. 13, Equation 7.5]) gives that $(-c)^2 + \sum(a_n^2 + b_n^2)/n^2 \leq 0$. The Theorem is proved.

*Remark.* The theorem of Freiling and Rinne which replaces the theorem of Schwarz and Cantor ([1, Appendix 2], [7, pp. 23 and 326], or [3]) that appears in the classical proof of the Theorem seems to avoid the maximum principle. However, Freiling and Rinne's present proof seems to require special properties of $\mathbf{R}^1$ that are not enjoyed by $\mathbf{R}^n$ for $n > 1$.

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