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A NOTE ON THE CATEGORY OF THE FREE LOOP SPACE

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ABSTRACT. A useful result in critical point theory is that the Ljusternik-Schnirelmann category of the space of *based* loops on a compact simply connected manifold M is infinite (because the cup length of M is infinite). However, the space of *free* loops on M may have trivial products. This note shows that, nevertheless, the space of the free loops also has infinite category.

1. INTRODUCTION

It is a standard result that if M is a simply connected compact manifold and $\Omega(M) = \Omega(M, x_0)$ is the space of *based* loops on M (based at x_0), then the Ljusternik-Schnirelmann category $\operatorname{cat}\Omega(M) = +\infty$. This follows from the now classical result (Serre [11], that the real (or rational) cohomology of $\Omega(M)$ has nontrivial cup products of arbitrary high length (see also [10]). An inspection of the proof will convince the reader that compactness is not required for the proof of this result. All that is required is that the real (or rational) cohomology $H^*(M)$ be finitely generated and for some $i > 0$ $H^i(M) \neq 0$. However, for the free loop space $\Lambda(M)$, where

$$\Lambda(M) = \{\alpha \in M^I,\ \alpha(0) = \alpha(1)\},$$

it isn't necessarily the case that the cohomology of $\Lambda(M)$ has nontrivial cup products. This is a relatively recent result of M. Vigué-Poirrier and D. Sullivan [13], where, for example, the reduced real cohomology of the free loops on the 2-sphere S^2 has trivial cup products. In view of this fact, it is natural to inquire about that category of the free loop space $\Lambda(M)$. We will show that when M satisfies the preceding conditions,

1. $\operatorname{cat}\Lambda(M) = +\infty$

2. $\Lambda(M)$ contains compact subsets C such that $\operatorname{cat}_{\Lambda(M)} C$ is arbitrarily large.

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This will allow a direct application of the Ljusternik-Schnirelmann method to, for example, functionals defined on the Sobolev space

$$W_T^{1,2} = \{f\colon [0,T] \to \mathbf{R}^n - 0,\ f(0) = f(T)\}.$$

See §4 and [1], [9].

The basic result is the following property of Hurewicz fibrations [7], which we prove in the next section.

Theorem. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ denote a Hurewicz fibration which admits a section $\sigma\colon B \to E$, and F, B, and E are 0-connected. If $Q \subset F$ is any subset of F, then*

$$\operatorname{cat}_F Q \le \operatorname{cat}_E Q.$$

When applied to the fibration $\Omega(M) \to \Lambda(M) \to M$ we obtain:

Corollary. *Let M denote a simply connected manifold (not necessarily compact) such that the real or rational cohomology $H^q(M)$ is finitely generated for each q and $H^i(M) \neq 0$ for some $i > 0$. Then $\operatorname{cat}\Lambda(M) = \infty$.*

§3 considers the category of the free loop space on configuration spaces.

2. Results

We recall first a basic lemma for Hurewicz fibrations [2]. If $F \xrightarrow{i} E \xrightarrow{p} B$ is a Hurewicz fibration, then there is a lifting function $\lambda\colon \Omega_p \to E^I$, where $\Omega_p = \{(x,\omega) \in E \times B^I : p(x) = \omega(0)\}$, where $\lambda(x,\omega)(0) = x$; $p(\lambda(x,\omega)(t)) = \omega(t)$, $0 \le t \le 1$. λ induces $\widetilde{\lambda}\colon E^I \to E^I$ by setting $\widetilde{\lambda}(\alpha) = \lambda[\alpha(0), p\alpha]$.

2.1. **Lemma** [7]. *$\widetilde{\lambda} \sim \mathrm{id}$ preserving projections, i.e., there is a homotopy Γ: $E^I \times I \to I$ such that $\Gamma_0 = \mathrm{id}$, $\Gamma_1 = \widetilde{\lambda}$ and $p\Gamma(\alpha,s)(t) = p\alpha(t)$ for $\alpha \in E^I$, s, $t \in I$.*

2.2. **Proposition.** *Let $F \xrightarrow{i} E \xrightarrow{p} B$ denote a Hurewicz fibration with base points $x_0 \in F$, $b_0 \in B$, $F = p^{-1}(b_0)$. We assume that $\Omega_p\colon \Omega(E,x_0) \to \Omega(B,b_0)$ admits a section σ. If Y is any space and $f\colon Y \to F$ is a map homotopic in E to the constant x_0, the f is homotopic in F to x_0.*

Proof. Let $A\colon Y \to E^I$ denote a homotopy such that $A(y)(0) = x_0$, $A(y)(1) = f(y)$. Consider the homotopy $\widehat{A}\colon Y \times I \to F$ given by

$$\widehat{A}(y,s) = \Gamma(A(y),s)(1), \quad 0 \le s \le 1, \quad y \in F.$$

Note that $\widehat{A}(y,0) = f(y)$, $\widehat{A}(y,s) \in F$ for $0 \le s \le 1$. Let $\widehat{A}(y,1) = g(y)$ so that $f \sim g\colon Y \to F$.

Now, define $C\colon Y \to E^I$ by $C(y) = \sigma p A(y)$ and observe that $pC(y) = pA(y)$. Now, define a homotopy $\widehat{C}\colon Y \times I \to F$ by

$$\widehat{C}(y,s) = \Gamma(C(y),s)(1), \quad 0 \le s \le 1, \quad y \in F.$$

Note that $\widehat{C}(y,0) = x_0$, $\widehat{C}(y,1) = g(y)$, and $\widehat{C}(y,s) \in F$, $0 \le s \le 1$. Thus, $f \sim x_0 \colon Y \to F$ and the proposition follows.

2.3. *Remark.* The reader will note the similarity between Proposition 2.2 and the classical result that $i_* \colon \pi_*(F) \to \pi_*(E)$ is injective when p admits a section [12]. Proposition 2.2 may also be thought of as a situation when the fiber is totally non-*homotopic* to zero.

Now we recall that a set Q in Y is categorical in Y if the inclusion $i \colon Q \longrightarrow Y$ is homotopic in Y to the constant map $y_0 \in Y$

2.4. **Corollary.** *Let $F \xrightarrow{i} E \xrightarrow{p} B$ denote a Hurewicz fibration as in 2.2 with E 0-connected. If U is categorical in E, then $U \cap F$ is categorical in F. Consequently, for any subset $Q \subset F$,*

$$\operatorname{cat}_F Q \le \operatorname{cat}_E Q.$$

Proof. The first part follows from Proposition 2.2, while the second part is immediate from the definition of category, namely, $\operatorname{cat}_Y X$ is the minimum number of categorical open sets in Y which cover X.

2.5. **Definition.** A manifold M will be called admissible if M is simply connected, the real (or rational) cohomology $H^*(M)$ is finitely generated and for some $i > 0$, $H^i(M) \neq 0$.

2.6. **Corollary.** *Let M denote a simply connected manifold with base point x_0. Then, we have the fibration with section*

$$\Omega(M, x_0) \xrightarrow{i} \Lambda(M) \underset{\sigma}{\overset{p}{\rightleftarrows}} M$$

where $p(\omega) = \omega(0)$ and $\sigma(x) = \tilde{x}$ the constant loop at $x \in M$. If $Q \subset \Omega(M, x_0)$ is an subset

$$\operatorname{cat}_{\Omega(M,x_0)} Q \le \operatorname{cat}_{\Lambda(M)} Q.$$

In particular, if M is admissible, then $\operatorname{cat} \Lambda(M) = \infty$.

2.7. *Remark.* Thus for example $\Lambda(S^2)$ has infinite category but trivial cup products over $\mathbb{R}$ [13].

2.8. *Remark.* Suppose F is a closed subset of E and $Q \subset F$. Then, the reverse inequality $\operatorname{cat}_E Q \le \operatorname{cat}_F Q$ holds whenever E is and ANR(normal). This is the case in Corollary 2.4 and hence there the inequality is actually an equality.

Now, we consider the question of compact subsets of $\Lambda(M)$ of arbitrarily high category.

2.9. **Lemma.** *Suppose X is a space such that for some field F the cup length of X over F using singular cohomology is $\ge k$; then X has a compact subset of category $> k$.*

Proof. We mention first that throughout we employ singular homology and cohomology with coefficients in F and will make use of the universal coefficient theorem *isomorphism*

$$\gamma\colon H^q(X) \longrightarrow \mathrm{Hom}_{\mathsf{F}}(H_q(X);\mathsf{F}).$$

Let $w = \alpha_1\alpha_2\cdots\alpha_k \in H^q(X)$ denote a nontrivial cup product of length k. Then, $\gamma(w) \neq 0$ and hence there is a singular cycle ζ such that $\gamma(w)([\zeta]) \neq 0$. Let A denote the (compact) support of ζ. Then, it is easy to check that if $i\colon A \to X$ is the inclusion map, $i^*(w) = i^*(\alpha_1)\cdots i^*(\alpha_k)$ is nonzero in $H^q(A)$. Thus the cup length of A in X is $\geq k$ and $\mathrm{cat}_X A > k$.

2.10. **Corollary.** *Let M denote an admissible manifold with base point x_0. Then, the space of based loops $\Omega(M, x_0)$ (and hence the space of free loops $\Lambda(M)$) contains compact subsets of arbitrarily high category.*

3. Configuration spaces

If M is any space the kth configuration space of X, $k \geq 1$, is defined by (see [6])

$$F_k(M) = \{(x_1, \dots, x_k), x_i \in M, x_i \neq x_j, \quad \text{for } i \neq j\}.$$

We will make of the following propositions. Cohomology will be over a field F of coefficients.

3.1. If M is a manifold (without boundary) and $k \geq 2$, then we have locally trivial fibrations

(i) $$F_{k-1}(M - Q) \to F_k(M) \xrightarrow{p} M$$

where $Q \in M$ and $p(x_1, \dots, x_k) = x_k$; and

(ii) $$(M - Q_{k-1}) \to F_k(M) \xrightarrow{q} F_{k-1}(M)$$

where $Q_{k-1} \subset M$ is a subset of $k-1$ elements and $q(x_1, \dots, x_k) = (x_1, \dots, x_{k-1})$.

3.2. If M is a simply connected manifold, $\dim M = m \geq 3$, and $H^i(M)$ is finitely generated over a field F for each i, then for $k \geq 1$ $F_k(M)$ is simply connected and $H^i(F_k(M))$ and $H^i(\Omega F_k(M))$ are finitely generated over F for each i.

We prove the proposition.

3.3. **Proposition.** *If M is a simply connected manifold,* $\dim M = m \geq 3$, *then for $k \geq 2$, the configuration space $F_k(M)$ is admissible.*

Proof. Because of 3.2 and the fact that $F_k(M)$ is finite dimensional, we need only show that for some $j > 0$, the real cohomology $H^j(F_k(M)) \neq 0$.

Case 1. $H^i(M) \neq 0$ for some $i \geq 1$. Choose i maximal so that $H^i(M) \neq 0$ and $0 \neq v \in H^i(M)$. We proceed by induction on k and employ the cohomology spectral sequence of the fibration (i) of 3.1. Choose $u \in F_{k-1}(M - Q)$ of maximal dimension so that $u \neq 0$. Then in the E_2-term of the spectral sequence $u \otimes v \neq 0$ and has dimension > 0. It is easy to see that $u \otimes v$ "survives" to E_∞ and contributes a nonzero element to $H^j(F_k(M))$, $j \geq 1$.

Case 2. $H^i(M) = 0$ for all $i > 0$. For $k = 2$ we employ the spectral sequence of the fibration (i) of 3.1 to see that $\mathbf{R} = H^{m-1}(M - Q) \approx H^{m-1}(F_2(M))$. For $k \geq 3$, we employ induction on k and the spectral sequence of the fibration (ii) of 3.1, together with the argument in Case 1 to obtain the desired result.

3.4. **Proposition.** *If M is a simply connected manifold, $\dim M \geq 3$, then for $k \geq 2$, $\operatorname{cat} \Lambda F_k(M) = \infty$ and $\Lambda F_k(M)$ contains compact subsets of arbitrarily high category.*

4. An application

In [9], Rabinowitz used the main result of §2 (corollary 2.10) in the special case where $M = \mathbf{R}^M - \{0\}$ to prove the existence of infinitely many periodic solutions of a certain Hamiltonian system. In this section we give an alternative argument for a key proposition in his treatment based upon a general abstract critical point theorem. That theorem is the analogue of a previous "linking" result in [5] done in the context of a relative cohomological equivariant index theory which will be replaced here by relative (Ljusternik-Schnirelmann) category theory introduced in [3] and [4].

We review first one version of relative category. If (E, A) is a topological pair with $A \neq \phi$ and closed in E, then for $A \subset X \subset E$ we define the relative category $\operatorname{cat}_E(X, A)$, as follows. A *categorical cover* of (X, A) consists of an open (in E) set $W \supset A$ and open sets $\{V_j\}$ such that

1. $W \cup (\cup V_j) \supset X$.
2. There is a homotopy of pairs $H\colon (W, A) \times I \to (E, A)$ such that $H_0(x) = x$ and $H_1(x) \in A$, $x \in A$.
3. Each V_j is contractible to a point in E.

4.1. **Definition.** $\operatorname{Cat}_E(X, A) = n$ if (X, A) admits a categorical cover $\{W, V_j\}$ with n sets V_j and n is minimal with this property. If no such finite categorical cover exists we set $\operatorname{cat}_E(X, A) = \infty$.

4.2. *Remark.* If $A = \phi$, $\operatorname{cat}_E(X, \phi) = \operatorname{cat}_E X$ has its usual meaning. The following properties are immediate:

4.3. **Proposition.**

(a) $A \subset X_1 \subset X_2$ *implies* $\operatorname{cat}_E(X_1, A) \leq \operatorname{cat}_E(X_2, A)$.
(b) $A \subset X_1$, $X_2 \subset E$ *implies* $\operatorname{cat}_E(X_1 \cup X_2, A) \leq \operatorname{cat}_E(X_1, A) + \operatorname{cat}_E X_2$.

Relative category may be used to define a "linking" concept as follows.

4.4. **Definition.** Let A and B denote disjoint closed sets in a space E. If

$$\operatorname{cat}_E(E-B, A) < \operatorname{cat}_E(E, A)$$

we say that A and B *link* (in the category sense). If, in addition, $\operatorname{cat}_E(E, A) = +\infty$, we say that A and B *strongly link.*

We review next a local form of the Palais-Smale condition $(\mathrm{PS})_S$. Let Λ denote an open set in a Banach space and $f\colon \Lambda \to \mathbf{R}$ a C^1-functional. f is said to satisfy $(\mathrm{PS})_s$ if any sequence $q_j \in \Lambda$ satisfying $f(q_j) \to s$ and $f(q_j) \to 0$ is precompact. $(\mathrm{PS})_s$ is used crucially in the following deformation theorem ([8], [9]). We will use that notation $K_c = \{q \in \Lambda, f'(c) = 0, \text{ and } f(q) = c\}$. Also $f^a = \{q \in \Lambda\colon f(q) < a\}$.

4.5. **Proposition.** *Let Λ denote an open set in a Banach space E and $f\colon \Lambda \to \mathbf{R}$ a C^1-functional. Suppose f satisfies $(\mathrm{PS})_s$ for all $s > \alpha$. Then, for any critical point $c > \alpha$, U a neighborhood of K_c, and $\bar{\varepsilon} > 0$, there is an $\varepsilon > 0$ and a deformation $\varphi\colon \Lambda \times I \to \Lambda$ such that*

(a) $\varphi_0 = \text{identity}$, $\varphi_t\colon \Lambda \to \Lambda$ *is a homeomorphism,* $t \in [0, 1]$.
(b) $\varphi(q, t) = q$ *if* $|f(q) - c| \geq \bar{\varepsilon}$.
(c) $\varphi(q, 1) \in f^{c-\varepsilon}$ *if* $q \in (f^{c+\varepsilon} - U)$.

If $K_c = \phi$, we may take $U = \phi$.

We may now state our abstract critical point theorem.

4.6. **Theorem.** *Let Λ denote an open set in a Hilbert (or Banach) space E such that Λ contains compact subsets of arbitrarily large category and let $f\colon \Lambda \to \mathbf{R}$ denote a C^1-functional. Suppose further that there are disjoint closed sets A and B in Λ such that*

1. $\operatorname{cat}_E(\Lambda - B, A) < \operatorname{cat}_E(\Lambda, A) = \infty$, *i.e. A and B strongly link,*
2. $\operatorname{cat}_E A < \infty$,
3. $\sup_A f < \inf_B f$,
4. *f is $(\mathrm{PS})_s$ for all $s > \sup_A f$;*

then f possesses an unbounded sequence of critical values.

Proof. For each integer $j \geq 0$ let

$$\Sigma_j = \{X | A \subset X \subset \Lambda,\ \operatorname{cat}_\Lambda(X, A) \geq j\}.$$

Observe that for each j, there is a compact set Y such that $\operatorname{cat}_E Y \geq j + \operatorname{cat}_E A$ and hence $\operatorname{cat}_E(A \cup Y, A) \geq j$. Thus, Σ_j is nonempty and f is bounded on $A \cup Y$. Hence, we may define

$$c_j = \inf_{X \in \Sigma_j} \sup_X f(x), \qquad j \geq 0$$

where

$$c_0 \leq c_1 \leq c_2 \leq \cdots c_j \leq c_{j+1} \leq \cdots.$$

Let $m = \operatorname{cat}_\Lambda(E-B, A)$. If $\operatorname{cat}_\Lambda(X, A) \geq m+1$, then $X \cap B \neq \phi$ for, otherwise $(X, A) \subset (\Lambda - B, B)$ and $\operatorname{cat}_\Lambda(X, A) \leq m$. Therefore,

$$\sup_A f = c_0 < c_{m+1},$$

i.e. a "jump" occurs at index $m+1$. We now show that each c_j, $j > m$, is a critical value. Let $c = c_j$, $j > m$ and consider K_c. Choose $\overline{\varepsilon} < \frac{1}{2}(c_{m+1} - c_0)$ and an open set $U \supset K_c$ such that $U \subset f^{-1}(c_{m+1} - \overline{\varepsilon}, \infty)$ and $\operatorname{cat}_\Lambda U = \operatorname{cat}_\Lambda K_c$. Now let $\varepsilon > 0$ and φ be the deformation given by Proposition 4.5. Observe that be φ remains fixed on A throughout the deformation. Take $X \in \Sigma_j$ so that $\operatorname{cat}_\Lambda(X, A) \geq j$ and $\sup_X f < c + \varepsilon$. Now

$$\operatorname{cat}_\Lambda(X, A) \leq \operatorname{cat}_\Lambda(X - U, A) + \operatorname{cat}_\Lambda(U).$$

If $K_c = \phi$, then $\operatorname{cat}_\Lambda(X, A) = \operatorname{cat}_\Lambda(X - U, A) \geq j$. On the other hand, $f(\varphi_1(X - U), A) < c - \varepsilon$. Since $\operatorname{cat}_\Lambda(\varphi_1(X - U), A) = \operatorname{cat}_\Lambda(X - U, A) \geq j$, this would force $c_j < c - \varepsilon = c_j - \varepsilon$, which is a contradiction. Thus $K_c \neq \phi$.

To show that the c_j are unbounded we proceed as follows. Let $\overline{c} = \sup c_j$. $\overline{c}$ is again a critical value. Let K denote the set of all critical points q such that $c_{m+1} \leq f(q) \leq \overline{c}$. The $(\mathrm{PS})_s$ condition for all $s > c_0$ forces K to be compact and $K \cap A = \phi$. Suppose $\operatorname{cat}_\Lambda K = k > 0$. Again choose $\overline{\varepsilon} < \frac{1}{2}(c_{m+1} - c_0)$ and an open set $U \supset K$ such that $U \subset f^{-1}(c_{m+1} - \overline{\varepsilon}, \infty)$ and $\operatorname{cat}_\Lambda U = k$. Furthermore, ε and φ will be as in Proposition 4.5, with $c = \overline{c}$. Choose an index j such that $c_j > \overline{c} - \varepsilon$ and $X \in \Sigma_{j+k}$ such that

$$\sup_X f < c_{j+k} + \varepsilon < \overline{c} + \varepsilon.$$

Then

$$\operatorname{cat}_\Lambda(X, A) \leq \operatorname{cat}_\Lambda(X - U, A) + \operatorname{cat}_\Lambda(U)$$

so that $\operatorname{cat}_\Lambda(X - U, A) \geq j$. But, then $\operatorname{cat}_\Lambda(\varphi_1(X - U), A) \geq j$ and $\varphi_1(X - U, A) \subset f^{\overline{c}-\varepsilon}$. This forces $c_j < \overline{c} - \varepsilon$, which is a contradiction.

As an application of Theorem 4.6, we give an alternative proof of a result of P. Rabinowitz, which he used to prove a slightly more general result [9]. The setting is the following Hamiltonian system.

$$\text{(HS)} \qquad \ddot{q} + V_q(t, q) = 0$$

where the potential function $V(t, q)$ satisfies the following conditions:

(V1) $V(t, q)$ is a C^1-function from $\mathbf{R} \times \Omega \to \mathbf{R}$, $\Omega = \mathbf{R}^n - \{0\}$, $n \geq 3$, which is T-periodic in t.

(V2) $V(t, q) < 0$ and $V(t, q) \to 0$, $V_q(t, q) \to 0$ as $|q| \to \infty$, uniformly in $t \in [0, T]$.

(V3) $V(t, q) \to -\infty$ as $q \to 0$, uniformly in $t \in [0, T]$.

(V4) There is a neighborhood N of 0 in $\mathbf{R}^n$ and a C^1-function $U\colon N - \{0\} \to \mathbf{R}$ such that $U(q) \to \infty$ as $q \to 0$ and $-V(t, q) \geq |U_q(q)|^2$ for $q \in N - \{0\}$ and all $t \in [0, T]$.

The period $T > 0$ will be fixed throughout the remainder of this section and $E_T = W_T^{1,2}(\mathbf{R}, \mathbf{R}^n)$ will denote the Sobolev space of T-periodic functions with square summable first derivatives, under the norm

$$\|q\| = (\int_0^T |\dot{q}|^2 \, dt + [q]^2)^{\frac{1}{2}}$$

where $\dot{q} = \frac{dq}{dt}$ and

$$[q] = \frac{1}{T} \int_0^T q(t) \, dt.$$

Set

$$\Lambda = \Lambda_T \{q \in E_T | q(t) \neq 0 \text{ for all } t \in [0, T]\}.$$

Λ is an open subset of E_T and Λ has the same homotopy type as the space $\Lambda(\mathbf{R}^n - 0)$ of free loops on $\mathbf{R}^n - 0$. $\mathbf{R}^n - 0$ is identified with the constant loops in $\mathbf{R}^n - 0$.

Corresponding to (HS) is the functional $I \colon \Lambda \to R$ given by

$$I(q) = \int_0^T (\frac{1}{2}|\dot{q}|^2 - V(t, q)) \, dt, \qquad q \in \Lambda.$$

Critical points of I give classical T-periodic solutions of (HS) (see [9]). We set

$$I^\varepsilon = \{q \in \Lambda \colon I(q) < \varepsilon\}.$$

4.7. **Proposition.** *Assuming* (VI)–(V3), *there is an* $\varepsilon > 0$ *and an* $R > 0$ *such that if* $B(0, R)$ *is the open ball of radius* R, $A = \mathbf{R}^n - B(0, R)$, $B = \Lambda - I^\varepsilon$, *then*

1. $\sup_A I < \varepsilon$
2. I^ε *is deformable into* A.
3. $\operatorname{cat}_{\Lambda_T}(E - B, A) = 0$, $\operatorname{cat}_{\Lambda_T}(\Lambda, A) = +\infty$.

Proof. First we choose a decreasing sequence $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$ and a corresponding increasing sequence $R_m > 0$ such that $R_m \to +\infty$ with the property that

$$|V(t, q)| < \varepsilon_m \text{ implies } |q| > R_m.$$

Choose an index k such that

$$R_k - [2\varepsilon_k]^{\frac{1}{2}} T > [2\varepsilon_k]^{\frac{1}{2}} T \tag{1}$$

and

$$-\int_0^T V(t, q) \, dt < \frac{1}{2} \varepsilon_k T, \quad \text{for } |q| \geq R_k \tag{2}$$

and set $\varepsilon = T e_k$, $R = R_k$, $A = \mathbf{R}^n - B(0, R)$, $B = \Lambda - I^\varepsilon$. If $q \in I^\varepsilon$,

$$\|\dot{q}\|_{L_2}^2 = \int_0^T |\dot{q}|^2 \leq 2I(q) \leq 2T\varepsilon_k \tag{3}$$

and hence

$$\|\dot{q}\|_{L_2} \leq (2\varepsilon)^{\frac{1}{2}}. \tag{4}$$

Now, write $q = [q] + Q$, where $Q(t) = q(t) = [q]$ and

$$[q] = \frac{1}{T}\int_0^T q(t)\,dt. \tag{5}$$

Recall that

$$\|Q\|_{L_\infty} = \max_t |Q(t)| \tag{6}$$

and the general inequality [9],

$$\|Q\|_\infty \leq T^{\frac{1}{2}}\|\dot{q}\|_{L_2}. \tag{7}$$

Hence,

$$\|Q\|_\infty \leq (2T\varepsilon)^{\frac{1}{2}}. \tag{8}$$

Now, consider a constant loop q, with $|q| \geq R$.

$$I(q) = -\int_0^T V(t, q)\,dt < \frac{\varepsilon}{2} < \varepsilon \tag{9}$$

and hence $A = \mathbf{R}^n - B(0, R) \subset I^\varepsilon$ and

$$\sup_A I < \varepsilon. \tag{10}$$

Consider now the homotopy, $H\colon I^\varepsilon \times [0, 1] \to \Lambda$, where

$$H(q, s) = [q] + (1 - s)Q, \qquad 0 \leq s \leq 1 \tag{11}$$

which is fixed on constant loops q. For $q \in I^\varepsilon$, it is easy to verify that

$$|[q]| \geq R - (2T\varepsilon)^{\frac{1}{2}} > (2\varepsilon T)^{\frac{1}{2}} \tag{12}$$

and using (8)

$$|Q(t)| \leq (2T\varepsilon)^{\frac{1}{2}} \text{ for all } t \in [0, 1]. \tag{13}$$

This forces

$$[q] + (1 - s)Q(t) \neq 0, \qquad 0 \leq s, \qquad t \leq 1. \tag{14}$$

Thus the homotopy has range in Λ and deforms I^ε into the subspace $\mathbf{R}^n - B(0, \rho)$, $\rho = (2\varepsilon T)^{1/2}$. If one follows H by a radial homotopy, we obtain a deformation of I^ε to $\mathbf{R}^n - B(0, R)$, with $\mathbf{R}_n - B(0, R)$ fixed throughout the composite homotopy. This also shows that $\mathrm{cat}_\Lambda(\Lambda - B, A) = 0$. Finally, since $\mathrm{cat}_\Lambda A = 2$ and $\mathrm{cat}_\Lambda = +\infty$, it is clear that $\mathrm{cat}_\Lambda(\Lambda, A) = +\infty$.

4.8. **Theorem.** (Rabinowitz[9]). *If we assume that* V *satisfies* (V1)-(V4), *the function I possesses an unbounded sequence of critical values.*

Proof. Rabinowitz [9] verifies that I satisfies $(\mathrm{PS})_s$ for all $s > 0$ and we will not repeat the argument. In view of Proposition 4.7, the proof is now a direct application of Theorem 4.6.

REFERENCES

1. A. Ambrosetti and V. Coti Zelati, *Critical points with lack of compactness and singular dynamical systems*, Ann. Mat. Pura Appl. (to appear).
2. E. Fadell, *On fiber spaces*, Trans. Amer. Math. Soc. **90** (1959), 1–14.
3. ——, *Cohomological methods in non-free G-spaces with applications to general Borsuk-Ulam theorems and critical point theorems for invariant functionals*, Nonlinear Functional Analysis and its Applications, Reidel, (1986), 1–47.
4. ——, *Lectures in cohomological index theories of G-spaces with applications to critical point theory*, Raccolta Di Universeta della Calabria, (1987), Cosenza, Italy.
5. E. Fadell and S. Husseini, *Relative cohomological index theories*, Adv. in Math. **64** (1987), 1–31.
6. E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1962), 111–118.
7. W. Hurewicz, *On the concept of fiber space*, Proc. Nat. Acad. Sci. **41** USA (1955), 956–961,
8. P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Report No. 65, 1984.
9. ——, *Periodic solutions for some forced singular Hamiltonian systems*, Festschrift fur Jurgen Moser (to appear).
10. J. T. Schwartz, *Generalizing the Lusternik-Schnirelmann theory of critical points*, Comm. Pure. Appl. Math. **17** (1964), 307–315.
11. J. P. Serre, *Homologie singulier des espaces fibrés*, Ann. of Math. **54** (1951), 425–505.
12. E. Spanier, *Algebraic topology*, McGraw-Hill, 1966.
13. M. Viguré-Poirrier and D. Sullivan, *The homology theory of the closed geodesic problem*, J. Differential Geom. **11** (1976), 633–644.

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