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REPRESENTATIVES FOR FINITE SETS

XING-DE JIA

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ABSTRACT. This paper considers a combinatorial problem by M. B. Nathanson [1], concerning simultaneous systems of representatives for two families of finite sets.

1. INTRODUCTION

Let $\mathcal{S} = \{S_i\}$ be a family of nonempty sets. The set X is a system of representatives for $\mathcal{S}$ if $X \cap S_i \neq \varnothing$ for every S_i in $\mathcal{S}$. If X is a system of representatives for $\mathcal{S}$ but no proper subset of X is a system of representatives for $\mathcal{S}$, then X is called a minimal system of representatives for $\mathcal{S}$. By $D(\mathcal{S})$ we denote the number of minimal systems of representatives for $\mathcal{S}$. Let $|S|$ denote the cardinality of the set S. If $\mathcal{S}$ consists of s pairwise disjoint sets S_i with $|S_i| = h$ for all i, then $D(\mathcal{S}) = h^s$.

Let $\mathcal{S} = \{S_i\}$ and $\mathcal{T} = \{T_j\}$ be two families of nonempty sets. A set X is called a simultaneous system of representatives for $\mathcal{S}$ and $\mathcal{T}$ if X is a minimal system of representatives for $\mathcal{S}$ and X is also a system of representatives for $\mathcal{T}$. $N(\mathcal{S},\mathcal{T})$ denotes the number of the simultaneous systems of representatives for $\mathcal{S}$ and $\mathcal{T}$. The study of the numbers $D(\mathcal{S})$ and $N(\mathcal{S},\mathcal{T})$ could be usefully applied to investigate minimal asymptotic bases in additive number theory [2].

In 1985, Nathanson [1] asked the following question:

Let $h \geq 2$ and $k \geq 1$. Does there exist a real number $\mu = \mu(h,k) \in (0,1)$ such that

$$N(\mathcal{S},\mathcal{T}) \leq D(\mathcal{S})\mu^t$$

holds for any families $\mathcal{S}$ and $\mathcal{T}$ of sets satisfying the following properties?

(i) $\mathcal{S} = \{S_i\}$ is a family of s nonempty, distinct sets S_i with $|S_i| \leq h$ for all i;
(ii) $\mathcal{T} = \{T_j\}$ is a family of t nonempty, pairwise disjoint sets T_j with $|T_j| \leq k$ for all j;
(iii) S_i is not a subset of T_j for all i and j.

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In this paper, it is proved that no such real number μ exists for any $h \geq 2$ and any $k \geq 1$. Adding some further restriction on $\mathcal{S}$, we prove that such μ exists in a special case.

2. Main results

Theorem 1. *Let $h \geq 2$ and $k \geq 1$. For any real number $\mu \in (0, 1)$, there exist two families of sets*

$$\mathcal{S} = \{S_i : i = 1, \ldots, s\} \quad \text{and} \quad \mathcal{T} = \{T_j : j = 1, \ldots, t\}$$

satisfying the following properties:

(i) $0 < |S_i| \leq h$ *for all* i;
(ii) $0 < |T_j| \leq k$ *for all* j;
(iii) $T_j \cap T_{j'} = \varnothing$ *for all* $j \neq j'$;
(iv) S_i *is not contained in* T_j *for all* i *and* j;
(v) $N(\mathcal{S}, \mathcal{T}) > D(\mathcal{S})\mu^t$.

Proof. Let μ be any real number so that $0 < \mu < 1$. Let t be an integer such that $\mu^t < 1/h$, and let $s = tk$. Let $a_1, \ldots, a_{h-1}$, $b_1, \ldots, b_s$ be $h - 1 + s$ different elements. Define

$$S_i = \{a_1, \ldots, a_{h-1}, b_i\}, \qquad T_j = \{b_{(j-1)k+1}, \ldots, b_{jk}\}$$

for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, t$. Let

$$\mathcal{S} = \{S_i : i = 1, 2, \ldots, s\}, \qquad \mathcal{T} = \{T_j : j = 1, 2, \ldots, t.\}$$

It is clear that

$$N(\mathcal{S}, \mathcal{T}) = 1, \qquad D(\mathcal{S}) = h - 1 + 1 = h.$$

Therefore we have

$$N(\mathcal{S}, \mathcal{T}) = 1 > h\mu^t = D(\mathcal{S})\mu^t,$$

which proves the theorem.

Theorem 1 means that the answer to the question is negative for any $h \geq 2$ and $k \geq 1$. However, we have the following result:

Theorem 2. *Let $h \geq 2$. Suppose*

(i) $\mathcal{S} = \{S_i\}$ *is a family of nonempty, distinct sets* S_i *with* $|S_i| \leq h$ *for all* i;
(ii) *Every* S_i *intersects at most one* S_j *in* $\mathcal{S}$ *other than* S_i *itself*;
(iii) $\mathcal{T} = \{T_j\}$ *is a family of* t *sets* T_j *with* $T_j = \{a_j\}$ *for all* j, *where the* a_j*'s are distinct elements*;
(iv) S_i *is not contained in* T_j *for any* i *and* j.

Then

$$N(\mathscr{S},\mathscr{T}) \le D(\mathscr{S})(1-1/h)^{t/2}. \tag{1}$$

Proof. By induction on t for any fixed s. If $t = 0$, then

$$N(\mathscr{S},\mathscr{T}) = D(\mathscr{S}),$$

hence (1) holds for $t = 0$ and any s. Let $t \ge 1$. Assume that (1) holds for any s and any $t' < t$.

Let

$$\mathscr{S} = \{S_i : i = 1, 2, \dots, s\} \quad \text{and} \quad \mathscr{T} = \{T_j : j = 1, 2, \dots, t\}$$

be two famililes of sets satisfying the conditions (i)–(iv). If there exists some $T_j = \{a_j\}$ such that $a_j \notin S_i$ for all i, then $N(\mathscr{S},\mathscr{T}) = 0$, hence (1) holds for t and any s. Now we assume that

$$S = \bigcup_{i=1}^{s} S_i \supseteq \{a_1, \dots, a_t\}.$$

We consider $T_t = \{a_t\}$. Then the following three cases may occur.

Case I. There exists an i' such that $a_t \in S_{i'}$, where $S_{i'} \cap S_i = \varnothing$ for all $i \ne i'$. $S_{i'} \nsubseteq T_t$ implies that $|S_{i'}| \ge 2$. It is readily verified that

$$\mathscr{S}' = \mathscr{S} \backslash \{S_{i'}\} \quad \text{and} \quad \mathscr{T}' = \{T_j : j = 1, \dots, t-1\}$$

satisfy the conditions (i)–(iv), and

$$D(\mathscr{S}) = |S_{i'}| D(\mathscr{S}').$$

If X is a simultaneous system of representatives for $\mathscr{S}$ and $\mathscr{T}$, then $X' = X \backslash \{a_t\}$ is a simultaneous system of representatives for $\mathscr{S}'$ and $\mathscr{T}'$. Conversely, if X' is a simultaneous system of representatives for $\mathscr{S}'$ and $\mathscr{T}'$, then $X = X' \cup \{a_t\}$ is a simultaneous system of representatives for $\mathscr{S}$ and $\mathscr{T}$. Therefore

$$\begin{aligned} N(\mathscr{S},\mathscr{T}) = N(\mathscr{S}',\mathscr{T}') &\le D(\mathscr{S}')(1-1/h)^{(t-1)/2} \\ &= (1/|S_{i'}|) D(\mathscr{S})(1-1/h)^{(t-1)/2} \\ &\le \tfrac{1}{2} D(\mathscr{S})(1-1/h)^{(t-1)/2} \\ &< D(\mathscr{S})(1-1/h)^{t/2}. \end{aligned}$$

Case II. There exists an i' such that $a_t \in S_{i'} \cap S_{i''}$ for some i''. It follows from (ii) that

$$(S_{i'} \cap S_{i''}) \cap S_i = \varnothing$$

for any $i \ne i'$ and $i \ne i''$. Let

$$|S_{i'} \cap S_{i''}| = r, \qquad |S_{i'} \backslash S_{i''}| = u, \qquad |S_{i''} \backslash S_{i'}| = v.$$

Since $a_t \in S_{i'} \cap S_{i''}$, it is clear that if X is a simultaneous system of representatives for $\mathscr{S}$ and $\mathscr{T}$, then $X \backslash \{a_t\}$ is a simultaneous system of representatives for

$$\mathscr{S}' = \mathscr{S} \backslash \{S_{i'}, S_{i''}\} \quad \text{and} \quad \mathscr{T}' = \{T_j : j = 1, 2, \ldots, t-1\}.$$

Conversely, if X' is a simultaneous system of representatives for $\mathscr{S}'$ and $\mathscr{T}'$, then $X = X' \cup \{a_t\}$ is a simultaneous system of representatives for $\mathscr{S}$ and $\mathscr{T}$. Hence

$$N(\mathscr{S}, \mathscr{T}) = N(\mathscr{S}', \mathscr{T}').$$

It is clear that $D(\mathscr{S}) = (r + uv)D(\mathscr{S}')$. (iv) implies that $r + u \geq 2$ and $r + v \geq 2$, thus $1/(r + uv) \leq 1 - 1/h$. Therefore

$$\begin{aligned} N(\mathscr{S}, \mathscr{T}) = N(\mathscr{S}', \mathscr{T}') &\leq D(\mathscr{S}')(1 - 1/h)^{(t-1)/2} \\ &= \frac{1}{r + uv} D(\mathscr{S})(1 - 1/h)^{(t-1)/2} \\ &\leq (1 - 1/h)D(\mathscr{S})(1 - 1/h)^{(t-1)/2} \\ &< D(\mathscr{S})(1 - 1/h)^{t/2} \end{aligned}$$

Case III. There exists an i' such that

$$a_t \in S_{i'} \backslash S_{i''} \qquad \text{and} \quad S_{i'} \cap S_{i''} \neq \varnothing$$

for some $i'' \neq i'$. Let

$$|S_{i'} \cap S_{i''}| = r, \qquad |S_{i'} \backslash S_{i''}| = u, \qquad |S_{i''} \backslash S_{i'}| = v.$$

It is clear that if there are two sets T_j such that $T_j \subseteq S_{i''} \backslash S_{i'}$, then $N(\mathscr{S}, \mathscr{T}) = 0$, hence (1) holds. If there exists exactly one $T_j = \{a_j\}$ such that $a_j \in S_{i''} \backslash S_{i'}$, then any simultaneous system X of representatives for $\mathscr{S}$ and $\mathscr{T}$ contains a_t and a_j. Hence X is a simultaneous system of representatives for $\mathscr{S}$ and $\mathscr{T}$ if and only if $X \backslash \{a_t, a_j\}$ is a simultaneous system of representatives for

$$\mathscr{S}' = \mathscr{S} \backslash \{S_{i'}, S_{i''}\} \quad \text{and} \quad \mathscr{T}' = \mathscr{T} \backslash \{T_{t'}, T_j\}.$$

It is easily seen that

$$D(\mathscr{S}) = (r + uv)D(\mathscr{S}').$$

Noticing that $r + uv \geq 2$, we have

$$\begin{aligned} N(\mathscr{S}, \mathscr{T}) = N(\mathscr{S}', \mathscr{T}') &\leq D(\mathscr{S}')(1 - 1/h)^{(t-2)/2} \\ &= \frac{1}{r + uv} D(\mathscr{S})(1 - 1/h)^{(t-2)/2} \\ &\leq \tfrac{1}{2} D(\mathscr{S})(1 - 1/h)^{(t-2)/2} \\ &= D(\mathscr{S})(1 - 1/h)^{t/2}. \end{aligned}$$

If there does not exist T_j such that $a_j \in S_{i''} \backslash S_{i'}$, i.e., if $(S_{i''} \backslash S_{i'}) \cap T_j = \varnothing$ for all j, then any simultaneous system X of representatives for $\mathscr{S}$ and $\mathscr{T}$

contains a_t and an element x of $S_{i''}\backslash S_{i'}$, hence $X\backslash\{a_t, x\}$ is a simultaneous system of representatives for

$$\mathscr{S}' = \backslash\{S_{i'}, S_{i''}\} \quad \text{and} \quad \mathscr{T}' = \{T_j : j = 1, 2, \dots, t-1\}.$$

Conversely, if X' is a simultaneous system of representatives for $\mathscr{S}'$ and $\mathscr{T}'$, then $X = X' \cup \{a_t, x\}$ is a simultaneous system of representatives for $\mathscr{S}$ and $\mathscr{T}$ for any x in $S_{i''}\backslash S_{i'}$. It follows the fact that $r \geq 1$, $0 \leq v \leq h-1$ and $u \geq 1$ that

$$\frac{v}{r+uv} \leq 1 - \frac{1}{h}.$$

Therefore

$$\begin{aligned} N(\mathscr{S}, \mathscr{T}) &= |S_{i''}\backslash S_{i'}| N(\mathscr{S}', \mathscr{T}') \\ &\leq V D(\mathscr{S}')(1 - 1/h)^{(t-1)/2} \\ &= \frac{v}{r+uv} D(\mathscr{S})(1 - 1/h)^{(t-1)/2} \\ &\leq (1 - 1/h) D(\mathscr{S})(1 - 1/h)^{(t-1)/2} \\ &< D(\mathscr{S})(1 - 1/h)^{t/2}. \end{aligned}$$

This completes the proof.

References

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Department of Mathematics, Graduate School and University Center, The City University of New York, New York, New York 10036