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IDEAL THEORETIC COMPLETE INTERSECTIONS IN P_K^3

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ABSTRACT. We describe the monomial curves in P_K^3 (K algebraically closed field of characteristic zero) that are set theoretical complete intersections on two binomial surfaces. We prove that they are exactly those which are ideal theoretic complete intersections. Using that, we get explicitly all monomial curves that are ideal theoretic complete intersections and a minimal generating basis for their ideals.

1. INTRODUCTION

In [M], T. T. Moh proves that all monomial curves in P_K^3 are set theoretic complete intersections on two binomial surfaces (i.e. if $I(C)$ denotes the ideal of a monomial curve C in $K[X_0, X_1, X_2, X_3]$, then there are binomials F and G in $K[X_0, X_1, X_2, X_3]$ such that $I(C) = \sqrt{(F, G)}$, provided that the characteristic of K is positive. In characteristic zero the case is completely different. For example, smooth monomial curves of a degree greater than four are not set theoretical complete intersections on any binomial surface (see [T]).

Here we will determine all monomial curves that are set theoretic complete intersections on two binomial surfaces in P_K^3, K of characteristic zero. Using this description we will be able to prove that these curves are exactly those monomial curves that are ideal theoretic complete intersections. Consequently, we also have a description of all the monomial curves that are ideal theoretic complete intersections and we explicitly give for each one of them a minimal generating basis for its ideal.

2.

Let K be an algebraically closed field of characteristic zero and C a monomial curve in P_K^3 with generic zero $(t^d, t^{a_1}u^{b_1}, t^{a_2}u^{b_2}, u^d)$ where d, a_1, a_2, b_1, b_2 are positive integers such that $a_1 \neq a_2$, $a_1 + b_1 = d$, $a_2 + b_2 = d$, and $\text{g.c.d.}(d, a_1, a_2) = 1$.

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According to [T], if a monomial space curve is a set theoretic complete intersection on a binomial surface, i.e. $I(C)=\sqrt{(F,G)}$ where F is a binomial in $K[X_0,X_1,X_2,X_3]$, the binomial has to be in one of the following types:

$$X_1^{n_1}-X_0^{n_0}X_2^{n_2}X_3^{n_3} \quad \text{or} \quad X_2^{m_2}-X_0^{m_0}X_1^{m_1}X_3^{m_3}$$

for appropriate nonnegative exponents.

We want to determine the monomial curves that are set theoretical complete intersections on two binomial surfaces. If both of the surfaces are of the same type then we get a contradiction by proving that there is a line in the intersection of them.

So we conclude that one of the two binomials should be of the form $X_1^{n_1}-X_0^{n_0}X_2^{n_2}X_3^{n_3}$ and the other one of the form $X_2^{m_2}-X_0^{m_0}X_1^{m_1}X_3^{m_3}$, for some nonnegative integers n_1, n_0, n_2, n_3, m_2, m_0, m_1, m_3 such that $n_1=n_0+n_2+n_3$, $m_2=m_0+m_1+m_3$, $\text{g.c.d.}(n_0,n_2,n_3)=1$ and $\text{g.c.d.}(m_0,m_1,m_3)=1$, since by [T] we can choose the binomials to be irreducible.

We claim that $n_2=0$ or $m_1=0$; otherwise if both of them are different from zero, the line $X_1=0$, $X_2=0$ belongs to the intersection, which is a contradiction.

Without loss of generality let us suppose that $m_1=0$. Then let $F=X_1^{n_1}-X_0^{n_0}X_2^{n_2}X_3^{n_3}$ and $G=X_2^{m_2}-X_0^{m_0}X_3^{m_3}$ and let $\varphi\colon K[X_0,X_1,X_2,X_3]\to K[X_0,X_2,X_3]$ given by $\varphi(X_0)=X_0^{n_1}$, $\varphi(X_1)=X_0^{n_0}X_2^{n_2}X_3^{n_3}$, $\varphi(X_2)=X_2^{n_1}$, and $\varphi(X_3)=X_3^{n_1}$. Then according to Theorem 2.4 of [T], we have $\sqrt{(\varphi(G))}=\sqrt{(X_2^d-X_0^{a_2}X_3^{b_2})}$. But $\varphi(G)=\varphi(X_2^{m_2}-X_0^{m_0}X_3^{m_3})=X_2^{m_2n_1}-X_0^{m_0n_1}X_3^{m_3n_1}$, and since characteristic of K is zero, $X_2^d-X_0^{a_2}X_3^{b_2}$ and $X_2^{m_2n_1}-X_0^{m_0n_1}X_3^{m_3n_1}$ have no multiple factors. We conclude that

$$X_2^{m_2n_1}-X_0^{m_0n_1}X_3^{m_3n_1}=X_2^d-X_0^{a_2}X_3^{b_2}.$$

So $m_2n_1=d$, $m_0n_1=a_2$, $m_3n_1=b_2$; and since $\text{g.c.d.}(m_0,m_1,m_3)=1$ (G irreducible), we have that $n_1=g_2=\text{g.c.d.}(d,a_2)$.

Then if $d=g_2d_2^*$, $a_2=g_2a_2^*$, $b_2=g_2b_2^*$, we have $m_2=d_2^*$, $m_0=a_2^*$, $m_3=b_2^*$; and so $G=X_2^{d_2^*}-X_0^{a_2^*}X_3^{b_2^*}$ and $F=X_1^{g_2}-X_0^{n_0}X_2^{n_2}X_3^{n_3}$. From F we have $g_2=n_0+n_2+n_3$ and $a_1g_2=dn_0+a_2n_2\Rightarrow a_1=d_2^*n_0+a_2^*n_2$.

Theorem 2.1. *Let C be a monomial curve in P_K^3 with generic zero $(t^d,t^{a_1}u^{b_1},t^{a_2}u^{b_2},u^d)$ and let $g_1=\text{g.c.d.}(a_1,d)$, $d=g_1d_1^*$, $a_1=g_1a_1^*$, $b_1=g_1b_1^*$, $g_2=\text{g.c.d.}(a_2,d)$, $d=g_2d_2^*$, $a_2=g_2a_2^*$, $b_2=g_2b_2^*$. Then C is set theoretic complete intersection on two binomial surfaces if and only if there exist nonnegative integers n_0, n_2 such that $n_0+n_2\le g_2$ and $a_1=d_2^*n_0+a_2^*n_2$ or there exist nonnegative integers m_0, m_1 such that $m_0+m_1\le g_1$ and $a_2=d_1^*m_0+a_1^*m_1$.*

Proof. $(\to)$ See above.

($\leftarrow$) Suppose that there exist integers n_0, n_2 such that $n_0 + n_2 \leq g_2$ and $a_1 = d_2^* n_0 + a_2^* n_2$. Let $n_3 = g_2 - n_0 - n_2$, $F = X_1^{g_2} - X_0^{n_0} X_2^{n_2} X_3^{n_3}$ and $G = X_2^{d_2^*} - X_0^{a_2^*} X_3^{b_2^*}$; then F is homogeneous and irreducible, since if $\mathrm{g.c.d.}(n_0, n_2, g_2) \neq 1$, from $a_1 = d_2^* n_0 + a_2^* n_2$ we get $\mathrm{g.c.d.}(a_1, g_2) = \mathrm{g.c.d.}(d, a_1, a_2) \neq 1$, a contradiction. Also G is homogeneous and irreducible. F and $G \in I(C)$ since $g_2 a_1 = d n_0 + a_2 n_2$ and $d_2^* a_2 = a_2^* d$. Let $\varphi\colon K[X_0, X_1, X_2, X_3] \to K[X_0, X_2, X_3]$ given by $\varphi(X_0) = X_0^{g_2}$, $\varphi(X_1) = X_0^{n_0} X_2^{n_2} X_3^{n_3}$, $\varphi(X_2) = X_2^{g_2}$, $\varphi(X_3) = X_3^{g_2}$, then $\varphi(G) = X_2^d - X_0^{a_2} X_3^{b_2}$; and so by Theorem 2.4 of [T] we conclude that C is a set theoretic complete intersection on F and G.

Example. Let m, n, i be positive integers such that $i \leq m$; then the curve C with generic zero $(t^{mn}, t^i u^{mn-i}, t^m u^{mn-m}, u^{mn})$ is a set theoretic complete intersection on the two binomials

$$F = x_1^m - x_2^i x_3^{m-i} \quad \text{and} \quad G = x_2^n - x_0 x_3^{n-1}.$$

It follows from Theorem 2.1 since $\mathrm{g.c.d.}(m, mn) = m$, $i = n0 + 1i$, and $m \geq i$.

Theorem 2.2. *For monomial space curves in P_K^3, the condition that C is an ideal theoretic complete intersection is equivalent to the condition that C is a set theoretic complete intersection on two binomial surfaces, if characteristic of K is zero.*

Proof. ($\rightarrow$) If C is an ideal theoretic complete intersection, then according to [B-R] or [B-S-V] we can find two binomials F, G such that $I(C) = (F, G) \Rightarrow I(C) = \sqrt{(F, G)}$; so C is a set theoretic complete intersection on two binomial surfaces.

($\leftarrow$) If C is a set theoretic complete intersection on two binomial surfaces, from Theorem 2.1, without loss of generality, we have that there exist two nonnegative integers n_0, n_2 such that $n_0 + n_2 \leq g_2$ and $a_1 = d_2^* n_0 + a_2^* n_2$. And so $I(C) = \sqrt{(F, G)}$, where $F = X_1^{g_2} - X_0^{n_0} X_2^{n_2} X_3^{n_3}$ and $G = X_2^{d_2^*} - X_0^{a_2^*} X_3^{b_2^*}$.

By [B-R] or [B-S-V] we have that the forms in a minimal basis of $I(C)$ can be chosen to be of the following types:

(1) There is exactly one form of the type $F_1 = X_1^{\alpha} - X_0^{\alpha_0} X_2^{\alpha_2} X_3^{\alpha_3}$ respectively $F_2 = X_2^{\beta} - X_0^{\beta_0} X_1^{\beta_1} X_3^{\beta_3}$, where $\deg F_1$ and $\deg F_2$ are minimum among all forms of those types.

(2) There are forms of the type $H_1 = X_0^{\gamma_0} X_3^{\gamma_3} - X_1^{\gamma_1} X_2^{\gamma_2}$, $\min \gamma_i > 0$, $\gamma_1 < \alpha$, and $\gamma_2 < \beta$.

(3) There are forms of the type $H_2 = X_0^{\delta_0} X_2^{\delta_2} - X_1^{\delta_1} X_3^{\delta_3}$, or $X_0^{\delta_0} X_1^{\delta_1} - X_2^{\delta_2} X_3^{\delta_3}$, $\min \delta_i > 0$, $\delta_1 < \alpha$, $\delta_2 < \beta$.

We have that $F = X_1^{g_2} - X_0^{n_0} X_2^{n_2} X_3^{n_3}$ is of F_1 type and $\deg F = g_2$ is minimum among forms of type F_1, since from $\alpha a_1 = \alpha_0 d + \alpha_2 a_2 \Rightarrow g_2 / \alpha a_1 \Rightarrow g_2 / \alpha$ since $\mathrm{g.c.d.}(g_2, a_1) = 1 \Rightarrow \alpha \geq g_2$.

$G = X_2^{d_2^*} - X_0^{a_2^*} X_3^{b_2^*}$ is of F_2 type and we claim $\deg G = d_2^*$ is minimum among forms of type $F_2 = X_2^{\beta} - X_0^{\beta_0} X_1^{\beta_1} X_3^{\beta_3}$.

From $\beta a_2 = \beta_0 d + \beta_1 a_1$ and $\text{g.c.d.}(d, a_2, a_1) = 1$, we have that $g_2/\beta_1 \Rightarrow \beta_1 = g_2 \beta_1^*$.

Then

$$\beta a_2 = \beta_0 d + \beta_1 a_1 = \beta_0 d + \beta_1(d_2^* n_0 + a_2^* n_2) = \beta_0 d + \beta_1^* d n_0 + \beta_1^* a_2 n_2$$
$$\Rightarrow a_2^*(\beta - \beta_1^* n_2) = d_2^*(\beta_0 + \beta_1^* n_0) \geq 0.$$

If $\beta_0 + \beta_1^* n_0 = 0$, we conclude that $\beta_0 = 0$, $n_0 = 0$, $\beta = \beta_1^* n_2$. Then from $\beta = \beta_0 + \beta_1 + \beta_3$ we get $\beta = \beta_1 + \beta_3$ and $\beta_3 \neq 0$ since $a_1 \neq a_2$. So $\beta_1^* n_2 = \beta_1 + \beta_3 \Rightarrow \beta_1^*/\beta_3 \Rightarrow \beta_3 = k\beta_1^*$, k a positive integer $\Rightarrow n_2 = g_2 + k$, a contradiction since $n_2 \leq g_2$.

So we have

$$a_2^*(\beta - \beta_1^* n_2) = d_2^*(\beta_0 + \beta_1^* n_0) > 0,$$

and since $\text{g.c.d.}(a_2^*, d_2^*) = 1 \Rightarrow d_2^*/\beta - \beta_1^* n_2 \Rightarrow$ there exists a positive integer ℓ such that $\beta - \beta_1^* n_2 = \ell d_2^* \Rightarrow \beta = \ell d_2^* + \beta_1^* n_2 \geq d_2^*$.

So we have proved that $F_1 = F$ and $F_2 = G$, and so $\alpha = g_2$ and $\beta = d_2^*$.

We claim that in $I(C)$ there are no forms of type H_1 or H_2. Suppose that there was a form of type $H_1 = X_0^{\gamma_0} X_3^{\gamma_3} - X_1^{\gamma_1} X_2^{\gamma_2}$, $\min \gamma_i > 0$, $\gamma_1 < g_2$, and $\gamma_2 < d_2^*$. Then from $\gamma_0 d = a_1\gamma_1 + a_2\gamma_2$ we have that $g_2/a_1\gamma_1$; but $\text{g.c.d.}(a_1, g_2) = 1$, so g_2/γ_1. This is a contradiction, since $\gamma_1 < g_2$.

Now suppose that there was a form of type $H_2 = X_0^{\delta_0} X_2^{\delta_2} - X_1^{\delta_1} X_3^{\delta_3}$, $\min \delta_i > 0$, $\delta_1 < g_2$, and $\delta_2 < d_2^*$. Then from $\delta_0 d + \delta_2 a_2 = \delta_1 a_1$ we have g_2/δ_1, a contradiction since $\delta_1 < g_2$.

And so we conclude that $I(C) = (F, G) = (X_1^{g_2} - X_0^{n_0} X_2^{n_2} X_3^{n_3}, X_2^{d_2^*} - X_0^{a_2^*} X_3^{b_2^*})$, which means that C is an ideal theoretic complete intersection. □

Corollary 2.3. *C is an ideal theoretic complete intersection if and only if there exist nonnegative integers n_0, n_2 such that $n_0 + n_2 \leq g_2$ and $a_1 = d_2^* n_0 + a_2^* n_2$ or there exist nonnegative integers m_0, m_1 such that $m_0 + m_1 \leq g_1$ and $a_2 = d_1^* m_0 + a_1^* m_1$. In the first case a minimal basis for $I(C)$ is $(X_1^{g_2} - X_0^{n_0} X_2^{n_2} X_3^{n_3}, X_2^{d_2^*} - X_0^{a_2^*} X_3^{b_2^*})$, and in the second case $(X_1^{d_1^*} - X_0^{a_1^*} X_3^{b_1^*}, X_2^{g_1} - X_0^{m_0} X_1^{m_1} X_3^{m_3})$.*

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