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CANONICAL RELATIVIZED CYLINDRIC SET ALGEBRAS

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ABSTRACT. For every suitable relational structure there is a canonical relativized cylindric set algebra. This construction is used to obtain a generalization of Resek's relative representation theorem, and a stronger version of the "Stone type representation theorem" by Andréka and Thompson.

§1. INTRODUCTION

Let $2 \leq n < \omega$. MGR_n is the set of n-ary merry-go-round identities:

$$\mathsf{s}^{\lambda}_{\kappa_1}\mathsf{s}^{\kappa_1}_{\kappa_2}\mathsf{s}^{\kappa_2}_{\kappa_3}\dots\mathsf{s}^{\kappa_{n-1}}_{\kappa_n}\mathsf{s}^{\kappa_n}_{\lambda}\mathsf{c}_{\lambda}x = \mathsf{s}^{\lambda}_{\kappa_n}\mathsf{s}^{\kappa_n}_{\kappa_1}\mathsf{s}^{\kappa_1}_{\kappa_2}\dots\mathsf{s}^{\kappa_{n-2}}_{\kappa_{n-1}}\mathsf{s}^{\kappa_{n-1}}_{\lambda}\mathsf{c}_{\lambda}x$$

where $\lambda, \kappa_1, \dots, \kappa_n$ are distinct ordinals. The merry-go-round identities are defined in D. Resek's dissertation [R75]. (See [R75, pp. 2–3, and 2.3.4, p. 34], or [HMT85, 3.2.88(1)], or [HR75, pp. 382–383]. For all other unexplained terminology and notation see [HMT71] or [HMT85].) One of the most significant results of [R75] is the *relative representation theorem*:

Theorem A ([R75, 5.23]). *Suppose $2 \leq \alpha < \omega$, $\mathfrak{A} \in \mathsf{CA}_\alpha$, $\mathfrak{A}$ is simple, complete, atomic, and satisfies MGR_κ for $2 \leq \kappa < \alpha$. Then $\mathfrak{A}$ is isomorphic to a relativized cylindric set algebra, i.e., $\mathfrak{A} \in \mathbf{I}\mathsf{RlCs}_\alpha$.*

Resek's proof of the relative representation theorem is very long, on the order of 100 typed pages. (The proof shows that $\mathfrak{A}$ must be complete, even though this is not explicitly assumed in the statement of Theorem 5.23 in [R75]. Hence "complete" should be inserted after "atomistic" in the statement of Theorem 4.3 in [HR75].)

Resek's relative representation theorem has the following consequence.

Theorem B ([R75, 5.27)]. *For every $\alpha \geq 2$ the following are equivalent*:

(i) $\mathfrak{A} \in \mathsf{CA}_\alpha \cap \mathbf{IS}\mathsf{RlCs}_\alpha$,
(ii) $\mathfrak{A} \in \mathsf{CA}_\alpha$ *and* $\mathfrak{A}$ *satisfies* MGR_κ *whenever* $2 \leq \kappa < \omega$.

Theorem B shows that $\mathsf{CA}_\alpha \cap \mathbf{IS}\mathsf{RlCs}_\alpha$ is finitely axiomatizable whenever α is finite, and countably schematizable ([HMT85, 4.1.4]) whenever α is infinite.

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Resek's proof of Theorem B from Theorem A requires only six pages, but makes use of two unpublished results of L. Henkin.

R. J. Thompson has improved Theorem B in two ways. First, $\{\mathrm{MGR}_n : 2 \leq n < \omega\}$ can be replaced by just MGR_2 and MGR_3. (See [HMT85, 3.2.88]. MGR_2 is 3.2.88(2), and MGR_3 is 3.2.88(3).) Second, CA_α can be replaced by the class NA_α, whose definition is obtained from that of CA_α by replacing postulate (C_4), $\mathsf{c}_\kappa \mathsf{c}_\lambda x = \mathsf{c}_\lambda \mathsf{c}_\kappa x$, of [HMT71, 1.1.1], by the weaker postulate (C_4^*), $\mathsf{c}_\kappa \mathsf{c}_\lambda x \geq \mathsf{c}_\lambda \mathsf{c}_\kappa x \cdot \mathsf{d}_{\lambda\mu}$ with $\mu \neq \kappa, \lambda$ (see [AT] or [T]). With these improvements in place, Theorem B is called the *Resek-Thompson theorem* in [AT]. Thompson's proof of the Resek-Thompson theorem is similar to Resek's and is also quite long. A short proof, due to H. Andréka, is presented in [AT].

Unfortunately, much of the power of the relative representation theorem is lost in the Resek-Thompson theorem. To see this, suppose $2 \leq \alpha < \omega$ and $\mathfrak{A}$ is a simple complete atomic CA_α satisfying MGR_κ for $2 \leq \kappa < \alpha$. Then the Resek-Thompson theorem implies that $\mathfrak{A}$ is a subalgebra of an algebra in $\mathbf{I}\mathsf{RICs}_\alpha$, but not that $\mathfrak{A}$ is itself already in $\mathbf{I}\mathsf{RICs}_\alpha$, as shown by the relative representation theorem. In particular, every simple finite CA_α satisfying the merry-go-round identities is isomorphic to a relativized cylindric set algebra, and is not merely embeddable in such an algebra. Clearly what is needed is a short proof of Resek's relative representation theorem with Thompson's improvements. Theorem C below achieves this goal and more: α can be infinite, $\mathfrak{A}$ need not be simple, and the representation is "canonical", in the sense that no arbitrary choices are made in its construction. In particular, the axiom of choice is not used.

Theorem C. *Suppose* $2 \leq \alpha$, $\mathfrak{A} \in \mathsf{NA}_\alpha$, $\mathfrak{A}$ *is complete, atomic, and satisfies* MGR_κ *for* $\kappa = 2, 3$. *Then* $\mathfrak{A} \cong \mathfrak{Rc}\mathfrak{At}\mathfrak{A} \in \mathsf{RICs}_\alpha$.

This theorem implies the Resek-Thompson theorem. For one direction it suffices to note that every algebra in $\mathbf{IS}\mathsf{RICs}_\alpha$ satisfies MGR_2 and MGR_3. For the other direction, suppose that $\mathfrak{A} \in \mathsf{NA}_\alpha$ and $\mathfrak{A}$ satisfies MGR_2 and MGR_3. Then $\mathfrak{A}$ has a complete and atomic extension $\mathfrak{A}' \in \mathsf{NA}_\alpha$ which also satisfies MGR_2 and MGR_3, by [HMT71, 2.7.5, 2.7.13]. By Theorem C, $\mathfrak{A}' \in \mathbf{I}\mathsf{RICs}_\alpha$, so $\mathfrak{A} \in \mathsf{NA}_\alpha \cap \mathbf{IS}\mathsf{RICs}_\alpha$.

The key idea in the proof of Theorem C is the construction of a canonical relativized cylindric set algebra $\mathfrak{Rc}\mathfrak{B}$ from any suitable relational structure $\mathfrak{B}$. (A precise definition of "suitable" is given in Definition 1 below.) For any such structure $\mathfrak{B}$, $\mathfrak{Rc}\mathfrak{B}$ is a complete and atomic relativized cylindric set algebra with unit element $V(\mathfrak{B})$ (Corollary 8) whose atoms are orbits of single sequences under a group of canonical permutations of the base $U(\mathfrak{B})$ (Lemmas 11 and 12). If $\mathfrak{A}$ is any atomic algebra which satisfies postulates 1.1.1(C_0)–(C_3), (C_5)–(C_7) of [HMT71], then the atom structure $\mathfrak{At}\mathfrak{A}$ is suitable. There is a canonical embedding $\mathsf{R}^{\mathfrak{A}}$ of $\mathfrak{A}$ into $\mathfrak{Rc}\mathfrak{At}\mathfrak{A}$ which is a Boolean isomorphism just in case $\mathfrak{A}$ is complete. The embedding preserves the diagonal elements and preserves cylindrification in one direction (Lemma 17). Cylindrification is fully preserved, and the canonical embedding is therefore an isomorphism, just in case $\mathfrak{A}$ also satisfies (C_4^*), MGR_2, and MGR_3.

Suitable structures arise naturally from certain sets of matrices of atoms of weakly associative relation algebras (defined in [M82]). If $\mathfrak{A}$ is an atomic weakly associative relation algebra, then $\mathfrak{A}\mathfrak{t}\mathfrak{C}\mathfrak{a}B_3\mathfrak{A}$ happens to be a suitable structure. (See [M89] for $\mathfrak{C}\mathfrak{a}$ and B_3.) The two-dimensional projection R of the ternary relation $V(\mathfrak{A}\mathfrak{t}\mathfrak{C}\mathfrak{a}B_3\mathfrak{A})$ was used in the first (unpublished) proof of Theorem 5.20 of [M82]. That proof did not appear in [M82] due to the difficulties encountered in trying to formulate a precise mathematical description of R. This paper finally overcomes those difficulties. (Also, the proof which *does* appear in [M82] yields a useful auxiliary result, namely Theorem 5.19.) I would like to thank Richard L. Kramer for many useful discussions in the early 1980s about R and how to describe it, and the referee for extensive help in clarifying the arguments and exposition of this paper.

§2. Definition of $\mathfrak{R}\mathfrak{c}\mathfrak{B}$

Definition 1. $\mathfrak{B}$ is a *suitable structure* if $\mathfrak{B} = \langle B, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ where α is a nonzero ordinal, $T_\kappa \subseteq B \times B$, $E_{\kappa\lambda} \subseteq B$, and the following conditions hold for all $\kappa, \lambda, \mu < \alpha$:

(i) T_κ is an equivalence relation on B,
(ii) $E_{\kappa\kappa} = B$,
(iii) $E_{\kappa\lambda} = T_\mu^*(E_{\kappa\mu} \cap E_{\mu\lambda})$ whenever $\kappa, \lambda \neq \mu$,
(iv) $T_\kappa \cap (E_{\kappa\lambda} \times E_{\kappa\lambda}) \subseteq Id$ whenever $\kappa \neq \lambda$.

Throughout this section we assume that $\mathfrak{B}$ is a suitable structure and $\alpha > 0$.

Lemma 2. *For all* $\kappa, \lambda, \mu < \alpha$,

(i) $E_{\kappa\lambda} = T_\mu^* E_{\kappa\lambda}$ *if* $\mu \neq \kappa, \lambda$,
(ii) $B = T_\kappa^* E_{\kappa\lambda}$,
(iii) $E_{\kappa\mu} \cap E_{\mu\lambda} \subseteq E_{\kappa\lambda}$,
(iv) $E_{\kappa\lambda} = E_{\lambda\kappa}$.

Proof. Parts (i)–(iii) are immediate consequences of Definition 1(i)–(iii). For part (iv), we may assume $\kappa \neq \lambda$. By Definition 1(i)–(iii) we have $E_{\kappa\lambda} \sim E_{\lambda\kappa} \subseteq T_\kappa^*(E_{\kappa\lambda} \sim E_{\lambda\kappa}) \cap T_\kappa^*(E_{\kappa\lambda} \cap E_{\lambda\kappa})$. But $T_\kappa^*(E_{\kappa\lambda} \sim E_{\lambda\kappa}) \cap T_\kappa^*(E_{\kappa\lambda} \cap E_{\lambda\kappa}) = \varnothing$ by Definition 1(i)(iv), so $E_{\kappa\lambda} \subseteq E_{\lambda\kappa}$. □

Definition 3. Let

$$Tr(\mathfrak{B}) = \bigcup_{n<\omega} \Big\{\langle x_0, \kappa_0, \dots, x_n, \kappa_n\rangle \in {}^{(n+1)}(B \times \alpha) : (\forall i < n)(x_i \neq x_{i+1} \wedge x_i \, T_{\kappa_i} \, x_{i+1})\Big\}.$$

The sequences in $Tr(\mathfrak{B})$ are called *trails* of $\mathfrak{B}$, or $\mathfrak{B}$-*trails*. Let $t = \langle x_0, \kappa_0, \dots, x_n, \kappa_n\rangle$ be a trail of $\mathfrak{B}$. We say that t *begins* at x_0, t *ends* at x_n, κ_n is the *pointer* of t, and t has *length* $|t| = n+1$. The trail t is *reduced* if the following conditions hold:

(i) if $1 = |t|$ and $x_0 \in E_{\kappa_0\lambda}$, then $\kappa_0 \le \lambda < \alpha$,
(ii) if $1 < |t|$ then $\kappa_{n-1} = \kappa_n$ and for all $\lambda < \alpha$, $x_n \in E_{\kappa_n\lambda}$ iff $\kappa_n = \lambda$,
(iii) if $0 \le i < |t| - 2$, then either $x_i \ne x_{i+2}$ or $\kappa_i \ne \kappa_{i+1}$.

Finally, for every $\lambda < \alpha$, let $t\lambda = \langle x_0, \kappa_0, \dots, x_{n-1}, \kappa_{n-1}, x_n, \lambda\rangle$.

All trails have length 1 or more. Condition (iii) applies only to trails of length 3 or more, and states that a reduced trail cannot have a subsequence of the form $\langle x, \lambda, y, \lambda, x\rangle$. If t is a trail, then so is $t\lambda$.

Definition 4. Let Q be the smallest equivalence relation on $Tr(\mathfrak{B})$ which contains all pairs of $\mathfrak{B}$-trails of the following three types:

(1) $\langle\langle x_0, \kappa_0, \dots, x_i, \lambda, y, \lambda, x_i, \kappa_i, \dots, x_n, \kappa_n\rangle$, $\langle x_0, \kappa_0, \dots, x_i, \kappa_i, \dots, x_n, \kappa_n\rangle\rangle$ where $0 \le i \le n$,
(2) $\langle\langle x_0, \kappa_0, \dots, x_n, \lambda, y, \kappa_n\rangle, \langle x_0, \kappa_0, \dots, x_n, \kappa_n\rangle\rangle$ where $\lambda \ne \kappa_n$,
(3) $\langle\langle x_0, \kappa_0, \dots, x_n, \lambda\rangle, \langle x_0, \kappa_0, \dots, x_n, \kappa_n\rangle\rangle$ where $x_n \in E_{\lambda\kappa_n}$.

For each $t \in Tr(\mathfrak{B})$, let $t^{\mathfrak{B}}$ be the Q-class of t, *i.e.*, $t^{\mathfrak{B}} = \{t' : t\,Q\,t'\}$. Let $U(\mathfrak{B}) = \{t^{\mathfrak{B}} : t \in Tr(\mathfrak{B})\}$. $U(\mathfrak{B})$ is called the *canonical base for* $\mathfrak{B}$, or $\mathfrak{B}$-*base*, and the equivalence classes in $U(\mathfrak{B})$ are called *canonical base points*.

The relations used to define Q are "reductions". A reduction of type (1) consists of the replacement of any subsequence of the form $\langle x, \lambda, y, \lambda, x\rangle$ by $\langle x\rangle$. Equivalent trails have the same beginnings, but may have different ends and different pointers, due to reductions of type (2) and (3).

The intuition behind trails and the definition of Q arises in the following way. Suppose that $\mathfrak{B}$ is a suitable structure and $\mathfrak{B}$ is already represented by a relativized cylindric set algebra with base U. Refer to the elements of U as "base points". Suppose also that R_x is the set of sequences in ${}^{\alpha}U$ associated with each $x \in B$. This representation must have certain properties. First, if $x \in E_{\kappa\lambda}$ then the κ-term of any sequence associated with x must be the same as the λ-term of that sequence, or, briefly, $p_\kappa = p_\lambda$ for every $p \in R_x$. Second, if $x\,T_\kappa\,y$, then every sequence $p \in R_x$ can have its κ-term altered to obtain a sequence $p' \in R_y$, hence $p'_\mu = p_\mu$ whenever $\mu \ne \kappa$. We wish to describe the representation R of $\mathfrak{B}$ using trails, and then use that description to construct a canonical representation entirely in terms of $\mathfrak{B}$-trails.

Let $t = \langle x, \kappa, \dots\rangle$ be a trail beginning at x. Then t may be conceived as instructions which can be applied to any sequence $p \in {}^{\alpha}U$ associated with x (but *not* applicable to any sequence not associated with x). The instructions in t select a particular base point, depending on p. It will turn out that t is reduced if there is no shorter trail which leads from p to the base point selected by t.

Some examples will show how the instructions should be followed. Let $p \in R_x$. The trail $\langle x, \kappa\rangle$ says, "Select the κ-term of the sequence." So, following

$\langle x, \kappa\rangle$ from p leads to the base point p_κ. The trail $\langle x, \kappa, y, \lambda\rangle$ says, "Alter the κ-term of the sequence to get a sequence associated with y, and then select the λ-term of the latter sequence." Following $\langle x, \kappa, y, \lambda\rangle$ from p leads to the base point p'_λ for some $p' \in R_y$ such that $p_\mu = p'_\mu$ whenever $\mu \neq \kappa$. Finally, the trail $\langle x, \kappa, y, \lambda, z, \nu\rangle$ says, "Alter the κ-term of the first sequence to get a sequence associated with y, then alter the λ-term of the latter sequence to get a third sequence associated with z, and then select the ν-term of the third sequence."

These examples show that the pointer has a different role from the other ordinals in a trail. The pointer tells which base point in the final sequence to select, while the other ordinals in the trail tell which term in one intermediate sequence should be altered to get the next sequence.

The instructions may be ambiguous. There may be more than one way to alter a sequence to get a satisfactory new sequence. In could happen, for example, that $p \in R_x$, $x T_\kappa y$, $p', p'' \in R_y$, $p_\mu = p'_\mu = p''_\mu$ whenever $\mu \neq \kappa$, and yet $p' \neq p''$. Following $\langle x, \kappa, y, \lambda\rangle$ from p could therefore lead to either p'_λ or p''_λ. We could, of course, make the instructions unambiguous by well-ordering the elements of U and always choosing the *first* suitable base point as the new κ-term. We could also complicate the structure of trails by adding ordinals to index *which* base point is being selected at each successive alteration. This procedure would considerably complicate the definition of $\mathfrak{Rc}\mathfrak{B}$. Nevertheless, the main result could be proved using such a construction.

Instead we assume that the representation has a special property which guarantees the instructions in every trail are unambiguous. More precisely, we assume that if $x, y \in \mathfrak{B}$, $p \in R_x$, $x T_\kappa y$, $p', p'' \in R_y$, and $p_\mu = p'_\mu = p''_\mu$ whenever $\mu \neq \kappa$, then $p' = p''$. We will derive the definition of Q under this assumption of unambiguity.

Consider the base points which can be reached from $p \in R_x$ by following trails which begin at x. Different trails may lead to the same base point. The definition of the equivalence relation Q is designed so that if two trails are equivalent via Q, then they lead to the same base point. This is why *canonical* base points are defined as Q-equivalence classes. Depending on the structure of the given representation, it may or may not be true that trails leading to the same base point are equivalent, but this *will* be true in the canonical representation.

To see the need for each of the three types of reductions in Q, consider the following example. Suppose $p \in R_x$, $x \neq y$, $p' \in R_y$, $x T_0 y$, $x, y \in E_{23}$, $p_\mu = p'_\mu$ whenever $\mu \neq 0$, and p_0, p_1, p_2, p'_0 are distinct. Following $\langle x, 0\rangle$ from p leads to p_0, while following $\langle x, 0, y, 0\rangle$ from p leads to p'_0, a base point distinct from p_0. On the other hand, following either $\langle x, 1\rangle$ or $\langle x, 0, y, 1\rangle$ from p leads to the same base point, since $p_1 = p'_1$ and, by the assumption of unambiguity, there can be no sequence in R_y other than p' which coincides with p at all terms except the 0-term. We must therefore define Q so that $\langle x,1\rangle Q\langle x,0,y,1\rangle$. This justifies the inclusion of reductions of type (2) in Q. Following either $\langle x,2\rangle$ or $\langle x,3\rangle$ from p also leads to the same base point, since $x\in E_{23}$ and hence $p_2=p_3$. By including reductions of type (1) in the definition of Q we accordingly get $\langle x,2\rangle Q\langle x,3\rangle$. For any κ, following $\langle x,0,y,0,x,\kappa\rangle$ from p requires first moving from p to p' (no other choice is available, by unambiguity), and then from p' to a sequence in R_x which differs from p' only in its 0-term. But there already is such a sequence, namely p. Hence, by unambiguity, the last sequence obtained by following $\langle x,0,y,0,x,\kappa\rangle$ from p must be p itself and the selected base point must be p_κ. Since this point is also selected by $\langle x,\kappa\rangle$, we must define Q so that $\langle x,0,y,0,x,\kappa\rangle Q\langle x,\kappa\rangle$. This is accomplished by including reductions of type (1). It turns out that no other reductions are needed. Even reductions of type (1) are not strictly necessary to construct a relative representation of a suitable structure. By including reductions of type (1) we guarantee that the resulting canonical relative representation is, in fact, unambiguous. Note that the reduced trails in this example are $\langle x,0\rangle$, $\langle x,1\rangle$, $\langle x,2\rangle$, and $\langle x,0,y,0\rangle$, which lead to p_0, p_1, p_2, and p_0', respectively.

In the next definition we give short names to the reductions and certain other relations. Then in the lemma we show that every canonical base point contains a unique reduced trail, and give an algorithm for computing the unique reduced trail in $t^{\mathfrak{B}}$ from t.

For any binary relation X, the transitive closure of X is $X^\omega = X\cup X^2\cup X^3\cup\dots$, where $X^2=X|X$, $X^3=X|X|X$, *etc.*, and $Do\,X$ is the domain of X. For any set U, Id_U is the identity relation on U.

Definition 5. For $n=1,2,3$, let P_n be the set of pairs of $\mathfrak{B}$-trails of type (n) in Definition 4. Let $P_4=P_3\cap\{\langle t\kappa,t\lambda\rangle:\kappa>\lambda\}$, $P_5=P_3|(P_1\cup P_2)|P_3$, $P_6=P_3\cup P_5$, $Z=Tr(\mathfrak{B})\sim Do\,(P_4\cup P_5)$, and $P_7=P_6^\omega|Id_Z$.

Lemma 6.

(i) P_2 *is a function, i.e.,* $P_2^{-1}|P_2\subseteq Id_{Tr(\mathfrak{B})}$,

(ii) P_3 *is an equivalence relation on* $Tr(\mathfrak{B})$,

(iii) $P_1^{-1}|P_1=Id_{Tr(\mathfrak{B})}\cup P_1|P_1^{-1}$,

(iv) $P_2^{-1}|P_1=P_2\cup P_1|P_2^{-1}$ *and* $P_1^{-1}|P_2=P_2^{-1}\cup P_2|P_1^{-1}$,

(v) $P_1^{-1}|P_3=P_3|P_1^{-1}$ *and* $P_3|P_1=P_1|P_3$,

(vi) $P_1^{-1}|P_3|P_1\subseteq P_5|P_5^{-1}\cup P_3$,

(vii) $P_1^{-1}|P_3|P_2\subseteq P_5|P_5^{-1}\cup P_5^{-1}$ *and* $P_2^{-1}|P_3|P_1\subseteq P_5|P_5^{-1}\cup P_5$,

(viii) $P_2^{-1}|P_3|P_2\subseteq P_3$,

(ix) $P_6^{-1}|P_6\subseteq P_6|P_6^{-1}$,

(x) $Q=(P_6\cup P_6^{-1})^\omega=P_6^\omega|(P_6^{-1})^\omega$,

(xi) *for every* $t\in Tr(\mathfrak{B})$, t *is reduced iff* $t\in Z$,

(xii) $Q=P_6^\omega|Id_Z|(P_6^{-1})^\omega=P_7|P_7^{-1}$,

(xiii) $Id_Z|P_7=Id_Z$,

(xiv) P_7 *is a function.*

Proof. Parts (i), (iii)–(v) follow just from the relevant parts of Definition 4. Part (ii) follows from Definition 1(ii) and Lemma 2(iii)(iv). Part (vi) follows from (ii), (iii), and (v). Part (vii) follows from (ii), (iv), and (v). Part (viii) follows from Definition 4 and Lemma 2(i). Parts (ii), (vi)–(viii) imply (ix).

We have $Id_{Tr(\mathfrak{B})} \subseteq P_1 \cup P_2 \cup P_3 \subseteq P_6 \subseteq Q$ by part (ii), and Q is the equivalence relation generated by $P_1 \cup P_2 \cup P_3$, so $Q = (P_6 \cup P_6^{-1})^\omega$. Part (ix) implies $(P_6 \cup P_6^{-1})^\omega \subseteq P_6^\omega|(P_6^{-1})^\omega$ by induction, and the opposite inclusion is trivially true. Thus (x) holds.

If $|t| = 1$ then t is reduced iff $t \notin Do\, P_4$. If $|t| > 1$ then t is reduced iff $t \notin Do\, P_5$. This follows from the observation that if the pointer can be changed by a type (3) reduction, then a type (2) reduction can be performed. More precisely, if $|t| > 1$, $t P_3 t'$, and $t \neq t'$, then $t' = t\lambda$ for some λ, and hence either $t \in Do\, P_2$ or $t' \in Do\, P_2$. Thus (xi) holds.

Let $R = P_4 \cup P_5$. Then $Tr(\mathfrak{B}) = Z \cup Do\, R$, so

$$\begin{aligned} Id_{Tr(\mathfrak{B})} &= Id_Z \cup Id_{Do\, R} \subseteq Id_Z \cup R|R^{-1} \\ &= Id_Z \cup R|Id_{Tr(\mathfrak{B})}|R^{-1} \subseteq Id_Z \cup R|(Id_Z \cup R|R^{-1})|R^{-1} \\ &= Id_Z \cup R|Id_Z|R^{-1} \cup R^2|(R^{-1})^2. \end{aligned}$$

Continuing in this way, we get

$$(1) \qquad Id_{Tr(\mathfrak{B})} = Id_Z \cup \bigcup_{0<k<n} R^k|Id_Z|(R^{-1})^k \cup R^n|(R^{-1})^n$$

whenever $2 \leq n < \omega$. If $t P_5 t'$ then $|t| > |t'|$, and if $t\kappa\, P_4\, t\lambda$ then $\kappa > \lambda$, so there are no infinite R-chains. Therefore, given an arbitrary $t \in Tr(\mathfrak{B})$, there is some n such that $\langle t, t\rangle \notin R^n|(R^{-1})^n$ and $2 \leq n < \omega$, which implies, by (1), that $\langle t, t\rangle \in Id_Z \cup \bigcup_{0<k<n} R^k|Id_Z|(R^{-1})^k$. This proves $Id_{Tr(\mathfrak{B})} \subseteq Id_Z \cup R^\omega|Id_Z|(R^{-1})^\omega$. Note that $Id_Z \cup R \subseteq P_6$. Consequently $Id_{Tr(\mathfrak{B})} \subseteq P_6^\omega|Id_Z|(P_6^{-1})^\omega$ and $P_6^\omega|P_6^\omega = P_6^\omega$. By (x), $Q = P_6^\omega|Id_{Tr(\mathfrak{B})}|(P_6^{-1})^\omega \subseteq P_6^\omega|P_6^\omega|Id_Z|(P_6^{-1})^\omega|(P_6^{-1})^\omega = P_6^\omega|Id_Z|(P_6^{-1})^\omega$. Note that $P_7^{-1} = Id_Z|(P_6^\omega)^{-1} = Id_Z|(P_6^{-1})^\omega$, so $Q = P_7|P_7^{-1}$. Thus (xii) holds.

By the relevant definitions, $Id_Z|P_6 = Id_Z|(Id_{Tr(\mathfrak{B})} \cup P_4 \cup P_4^{-1} \cup P_5) = Id_Z \cup Id_Z|P_4^{-1}$. Also, $P_4^{-1}|P_6 \subseteq P_3|P_6 = P_6$ by part (ii). Consequently $Id_Z|P_6^\omega = Id_Z \cup Id_Z|P_4^{-1}$ by induction. Also, $P_4^{-1}|Id_Z = \varnothing$, so $Id_Z|P_7 = Id_Z|P_6^\omega|Id_Z = (Id_Z \cup Id_Z|P_4^{-1})|Id_Z = Id_Z$. Thus (xiii) holds.

By the definition of P_7, (x), (xii), and (xiii), $P_7^{-1}|P_7 = Id_Z|(P_6^\omega)^{-1}|P_6^\omega|Id_Z \subseteq Id_Z|Q|Id_Z = Id_Z|P_7|P_7^{-1}|Id_Z = Id_Z$, so (xiv) holds. □

By Lemma 6, P_7 is a function contained in Q, $P_7(t)$ is reduced, and t is reduced just in case $P_7(t) = t$. Thus P_7 maps each trail t to the unique reduced trail in $t^{\mathfrak{B}}$. Since P_3 is an equivalence relation and $P_3|P_5 = P_5 = P_5|P_3$, we have $P_6^\omega = (P_3 \cup P_5)^\omega = P_3 \cup P_5^\omega$, so $P_7 = P_6^\omega | Id_Z = (P_3 \cup P_5^\omega)|Id_Z = (Id_{Tr(\mathfrak{B})} \cup P_4 \cup P_4^{-1} \cup P_5^\omega)|Id_Z = Id_Z \cup (P_4 \cup P_5^\omega)|Id_Z$. In other words, to compute $P_7(t)$ in case t is not reduced, follow a P_5-chain until a trail is obtained which is not in $Do\,P_5$, and then reduce the pointer as much as possible by using P_4 once. Notice that $|P_7(t)| \leq |t|$ for every trail t. If $|t| = 1$, then $t \notin Do\,P_5$, and either t is already reduced, or else $t\,P_4\,P_7(t)$, where $P_7(t) = t\kappa$ for some ordinal κ which is smaller than the pointer of t.

Definition 7. For every $x \in B$, $\mathsf{R}_x^{\mathfrak{B}} = \{\langle (t\kappa)^{\mathfrak{B}} : \kappa < \alpha \rangle : t \in Tr(\mathfrak{B}),\ t$ ends at $x\}$. Let $V(\mathfrak{B}) = \bigcup_{x \in B} \mathsf{R}_x^{\mathfrak{B}}$ and $\mathfrak{Rc}\mathfrak{B} = \mathfrak{Rl}_{V(\mathfrak{B})}\mathfrak{A}$, where $\mathfrak{A}$ is the subalgebra of $\mathfrak{Sb}\,{}^\alpha U(\mathfrak{B})$ which is completely generated by $\{\mathsf{R}_x^{\mathfrak{B}} : x \in B\}$. $\mathfrak{Rc}\mathfrak{B}$ is the *canonical relativized cylindric set algebra* of $\mathfrak{B}$.

Corollary 8. *$\mathfrak{Rc}\mathfrak{B}$ is a complete atomic relativized cylindric set algebra.*

It turns out that $\mathsf{R}_x^{\mathfrak{B}}$ is an atom in $\mathfrak{Rc}\mathfrak{B}$, for every $x \in B$. To prove this we will use the next lemma.

For every set U, if p is a one-to-one function from U into U, then $\tilde{p}$ is the function which maps $Sb({}^\alpha U)$ to $Sb({}^\alpha U)$ and is defined by $\tilde{p}X = \{a \in {}^\alpha U : a|p^{-1} \in X\}$ for every $X \subseteq {}^\alpha U$. (See [HMT85, p. 15].) It follows that if p is a permutation of U, then $\tilde{p}X = \{a|p : a \in X\}$.

Lemma 9. *Suppose U and B are sets, $R_x \subseteq {}^\alpha U$ for every $x \in B$, and P is a set of permutations of U. Assume, for every $x \in B$, that P preserves R_x, i.e., $(\forall p \in P)\,(\tilde{p}R_x = R_x)$, and that P acts transitively on R_x, i.e., $(\forall a, b \in R_x)\,(\exists p \in P)\,(a|p = b)$. Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{Sb}\,{}^\alpha U$ which is completely generated by $\{R_x : x \in B\}$. Then $\{R_x : x \in B\} \subseteq At\mathfrak{A}$.*

Proof. P preserves $\{R_x : x \in B\}$, and therefore preserves all relations S in $\mathfrak{A}$. If some such relation S were properly contained within some R_x, then it would be possible, by the transitivity of P on R_x, to pick a $p \in P$ which moves some sequence from S to $R_x \sim S$, contradicting the fact that every $p \in P$ must preserve S. □

Definition 10. For every $\mathfrak{B}$-trail $s = \langle x_0, \kappa_0, x_1, \kappa_1, \ldots, x_{n-1}, \kappa_{n-1}, x_n, \kappa_n \rangle$ let $\check{s} = \langle x_n, \kappa_{n-1}, x_{n-1}, \kappa_{n-2}, \ldots, x_1, \kappa_0, x_0, \kappa_n \rangle$. If $t = \langle y_0, \lambda_0, \ldots, y_m, \lambda_m \rangle$ is any other $\mathfrak{B}$-trail, then $s \odot t$ is defined if s ends where t begins, in which case

$$s \odot t = \langle x_0, \kappa_0, x_1, \kappa_1, \ldots, x_{n-1}, \kappa_{n-1}, y_0, \lambda_0, \ldots, y_m, \lambda_m \rangle .$$

Also,

$$L_s(t) = \begin{cases} s \odot t & \text{if } x_n = y_0 \\ \check{s} \odot t & \text{if } x_n \neq y_0 = x_0 \\ t & \text{if } x_n \neq y_0 \neq x_0 \end{cases},$$

and, for any $X \subseteq Tr(\mathfrak{B})$, $l_s(X) = \bigcup_{t \in X} L_s(t)^{\mathfrak{B}}$. Let

$$Pm(\mathfrak{B}) = \{ l_s : s \in Tr(\mathfrak{B}) \} .$$

If s and t are $\mathfrak{B}$-trails, then $\breve{s}$ is a $\mathfrak{B}$-trail of the same length, and if $s \odot t$ is defined, then $s \odot t$ is also a $\mathfrak{B}$-trail with $|s \odot t| = |s| + |t| - 1$. The associative law for $\odot$ holds whenever both sides are defined. If $|s| = 1$, then $\breve{s} = s$ and $L_s(t) = t$ for every $\mathfrak{B}$-trail t. Note that $(s \odot t)\kappa = s \odot t\kappa$ and $L_s(t)\kappa = L_s(t\kappa)$.

Lemma 11. *Suppose* $s, t \in Tr(\mathfrak{B})$. *Then*

(i) $l_s(t^{\mathfrak{B}}) = L_s(t)^{\mathfrak{B}}$,
(ii) $t^{\mathfrak{B}} = l_{\breve{s}} l_s(t^{\mathfrak{B}}) = l_s l_{\breve{s}}(t^{\mathfrak{B}})$,
(iii) l_s *is a permutation of* $U(\mathfrak{B})$ *and* $(l_s)^{-1} = l_{\breve{s}}$.

Proof. Suppose s begins at x_0 and ends at x_n. If t begins at y_0 then $L_s(t)$ begins at $[x_0, x_n](y_0)$, where $[x_0, x_n]$ is the permutation of B which interchanges x_0 and x_n and fixes all other elements of B. Hence, if $L_s(t)$ and $L_s(t')$ have the same beginning, then so do t and t', and the definitions of $L_s(t)$ and $L_s(t')$ fall into the same cases, *i.e.*, $L_s(t) = t$ iff $L_s(t') = t'$, $L_s(t) = s \odot t$ iff $L_s(t') = s \odot t'$, and $L_s(t) = \breve{s} \odot t$ iff $L_s(t') = \breve{s} \odot t'$. It follows easily that L_s is one-to-one. Furthermore, t, $L_{\breve{s}}L_s(t)$, and $L_s L_{\breve{s}}(t)$ have the same beginning.

According to the definitions of $\odot$ and L_s, t is always a final segment of $L_s(t)$. Consequently L_s preserves P_1, P_2, and P_3, *i.e.*, $L_s^{-1}|P_1|L_s \subseteq P_1$, $L_s^{-1}|P_2|L_s \subseteq P_2$, and $L_s^{-1}|P_3|L_s \subseteq P_3$. So L_s also preserves Q, *i.e.*, $L_s^{-1}|Q|L_s \subseteq Q$.

To prove (i), first suppose $t' \in l_s(t^{\mathfrak{B}})$. Then there is some $t'' \in t^{\mathfrak{B}}$ such that $t' \in L_s(t'')^{\mathfrak{B}}$. Therefore $t'' Q t$ and $t' Q L_s(t'')$. Since L_s preserves Q, we have $L_s(t'') Q L_s(t)$, so $t' Q L_s(t)$, *i.e.*, $t' \in L_s(t)^{\mathfrak{B}}$. Thus $l_s(t^{\mathfrak{B}}) \subseteq L_s(t)^{\mathfrak{B}}$. The opposite inclusion holds trivially.

For the proof of (ii), first check that $L_{\breve{s}}L_s(t)$ is either t, $\breve{s} \odot s \odot t$, or $s \odot \breve{s} \odot t$. We will show $L_{\breve{s}}L_s(t) \in t^{\mathfrak{B}}$ by considering three cases. Obviously $L_{\breve{s}}L_s(t) \in t^{\mathfrak{B}}$ if $L_{\breve{s}}L_s(t) = t$. Suppose $L_{\breve{s}}L_s(t) = \breve{s} \odot s \odot t \neq t$. Then $|s| > 1$ and $\breve{s} \odot s \odot t \in Do\, P_1$. In fact, $\breve{s} \odot s \odot t\, P_1^{|s|-1}\, t$, so $L_{\breve{s}}L_s(t) \in t^{\mathfrak{B}}$. Similarly, if $L_{\breve{s}}L_s(t) = s \odot \breve{s} \odot t \neq t$, then $s \odot \breve{s} \odot t\, P_1^{|s|-1}\, t$, so $L_{\breve{s}}L_s(t) \in t^{\mathfrak{B}}$. From $L_{\breve{s}}L_s(t) \in t^{\mathfrak{B}}$ and part (i) we get $t^{\mathfrak{B}} = (L_{\breve{s}}L_s(t))^{\mathfrak{B}} = l_{\breve{s}}(L_s(t)^{\mathfrak{B}}) = l_{\breve{s}} l_s(t^{\mathfrak{B}})$, which shows one equality of (ii). The other equality holds similarly.

Finally, part (iii) follows from parts (i) and (ii). □

By Lemma 11, $Pm(\mathfrak{B})$ is a group of permutations of the canonical base points. Hence the functions in $Pm(\mathfrak{B})$ are called *canonical permutations* of $U(\mathfrak{B})$.

Lemma 12. *Let* $x \in B$. *Then*

(i) $Pm(\mathfrak{B})$ *preserves* $\mathsf{R}_x^{\mathfrak{B}}$, *and* $Pm(\mathfrak{B})$ *acts transitively on* $\mathsf{R}_x^{\mathfrak{B}}$.
(ii) $\mathsf{R}_x^{\mathfrak{B}} \in At\mathfrak{Rc}\mathfrak{B}$.

Proof. By Lemma 9, (i) implies (ii). To prove (i), we first show that $\tilde{l}_s \mathsf{R}_x^{\mathfrak{B}} \subseteq \mathsf{R}_x^{\mathfrak{B}}$ for every $s \in Pm(\mathfrak{B})$ using Lemma 11(i):

$$\begin{aligned}
\tilde{l}_s \mathsf{R}_x^{\mathfrak{B}} &= \left\{ a|l_s \colon a \in \{\langle (t\kappa)^{\mathfrak{B}} \colon \kappa < \alpha\rangle \colon t \in Tr(\mathfrak{B}),\ t \text{ ends at } x\} \right\} \\
&= \left\{ \langle (t\kappa)^{\mathfrak{B}} \colon \kappa < \alpha\rangle | l_s \colon t \in Tr(\mathfrak{B}),\ t \text{ ends at } x \right\} \\
&= \left\{ \langle l_s((t\kappa)^{\mathfrak{B}}) \colon \kappa < \alpha\rangle \colon t \in Tr(\mathfrak{B}),\ t \text{ ends at } x \right\} \\
&= \left\{ \langle L_s(t\kappa)^{\mathfrak{B}} \colon \kappa < \alpha\rangle \colon t \in Tr(\mathfrak{B}),\ t \text{ ends at } x \right\} \\
&= \left\{ \langle (L_s(t)\kappa)^{\mathfrak{B}} \colon \kappa < \alpha\rangle \colon t \in Tr(\mathfrak{B}),\ t \text{ ends at } x \right\} \\
&\subseteq \mathsf{R}_x^{\mathfrak{B}} .
\end{aligned}$$

Since l_s and $l_{\breve{s}}$ are inverses, so are $\tilde{l}_s$ and $\tilde{l}_{\breve{s}}$. Hence $\mathsf{R}_x^{\mathfrak{B}} = \tilde{l}_{\breve{s}}\tilde{l}_s \mathsf{R}_x^{\mathfrak{B}} \subseteq \tilde{l}_s \mathsf{R}_x^{\mathfrak{B}} \subseteq \mathsf{R}_x^{\mathfrak{B}}$, so $\mathsf{R}_x^{\mathfrak{B}}$ is preserved by l_s.

To show that $Pm(\mathfrak{B})$ acts transitively on $\mathsf{R}_x^{\mathfrak{B}}$, let us assume that t and t' are trails which end at x. We wish to find a canonical permutation l_s which maps $\langle (t\kappa)^{\mathfrak{B}} : \kappa < \alpha\rangle$ to $\langle (t'\kappa)^{\mathfrak{B}} : \kappa < \alpha\rangle$. Let $s = t' \odot \breve{t}$. Note that s is defined since t' ends where $\breve{t}$ begins. Let $\kappa < \alpha$. Then $L_s(t\kappa) = t' \odot \breve{t} \odot t\kappa$. If $|t| = 1$, then $t' \odot \breve{t} \odot t\kappa = t'\kappa$, so $L_s(t\kappa) \in (t'\kappa)^{\mathfrak{B}}$, and if $|t| > 1$, then $t' \odot \breve{t} \odot t\kappa\, P_1^{|t|-1}\, t'\kappa$, and again $L_s(t\kappa) \in (t'\kappa)^{\mathfrak{B}}$. Hence $\langle (t\kappa)^{\mathfrak{B}} : \kappa < \alpha\rangle | l_s = \langle l_s(t\kappa^{\mathfrak{B}}) : \kappa < \alpha\rangle = \langle (L_s(t\kappa))^{\mathfrak{B}} : \kappa < \alpha\rangle = \langle (t'\kappa)^{\mathfrak{B}} : \kappa < \alpha\rangle$. □

By Lemma 12(i), if $a \in \mathsf{R}_x^{\mathfrak{B}}$, then $\mathsf{R}_x^{\mathfrak{B}} = \{a|p \colon p \in Pm(\mathfrak{B})\}$, *i.e.*, every $\mathsf{R}_x^{\mathfrak{B}}$ is the orbit of a single sequence under the group $Pm(\mathfrak{B})$.

Lemma 13. *For every* $x \in B$ *and* $\kappa, \lambda < \alpha$, $x \in E_{\kappa\lambda}$ *iff* $\mathsf{R}_x^{\mathfrak{B}} \subseteq \mathsf{D}_{\kappa\lambda}^{[V(\mathfrak{B})]}$.

Proof. If $\kappa = \lambda$ then the result holds by Definition 1(ii), so assume $\kappa \neq \lambda$ and $x \in E_{\kappa\lambda}$. Let $\langle (t\mu)^{\mathfrak{B}} \colon \mu < \alpha\rangle \in \mathsf{R}_x^{\mathfrak{B}}$. Then t ends at x and $t\kappa\, P_3\, t\lambda$, so $(t\kappa)^{\mathfrak{B}} = (t\lambda)^{\mathfrak{B}}$, which implies $\langle (t\mu)^{\mathfrak{B}} \colon \mu < \alpha\rangle \in \mathsf{D}_{\kappa\lambda}^{[V(\mathfrak{B})]}$.

Now assume $\mathsf{R}_x^{\mathfrak{B}} \subseteq \mathsf{D}_{\kappa\lambda}^{[V(\mathfrak{B})]}$. Let $t = \langle x, \kappa\rangle$. Then $\langle (t\mu)^{\mathfrak{B}} \colon \mu < \alpha\rangle \in \mathsf{R}_x^{\mathfrak{B}}$, so $t\,Q\,t\lambda$. But t and $t\lambda$ are not in the domain of P_5 since $|t| = |t\lambda| = 1$, so $t\,P_3\,t\lambda$, *i.e.*, $x \in E_{\kappa\lambda}$. □

Lemma 14. *If* $x, y \in B$, $\mu < \alpha$, *and* $x\,T_\mu\,y$, *then* $\mathsf{R}_x^{\mathfrak{B}} \subseteq \mathsf{C}_\mu^{[V(\mathfrak{B})]}\mathsf{R}_y^{\mathfrak{B}}$.

Proof. If $x = y$, then $\mathsf{R}_x^{\mathfrak{B}} = \mathsf{R}_y^{\mathfrak{B}} \subseteq \mathsf{C}_\mu^{[V(\mathfrak{B})]}\mathsf{R}_y^{\mathfrak{B}}$, so we may assume $x \neq y$. Suppose $\langle (t\kappa)^{\mathfrak{B}} : \kappa < \alpha\rangle \in \mathsf{R}_x^{\mathfrak{B}}$. Let $t' = t \odot \langle x, \mu, y, \mu\rangle$. Note that $\langle (t'\kappa)^{\mathfrak{B}} : \kappa < \alpha\rangle \in \mathsf{R}_y^{\mathfrak{B}}$. If $\kappa \neq \mu$ then $t\kappa = (t \odot \langle x, \kappa\rangle)\, Q\, (t \odot \langle x, \mu, y, \kappa\rangle) = t'\kappa$. So $(t\kappa)^{\mathfrak{B}} = (t'\kappa)^{\mathfrak{B}}$ whenever $\kappa \neq \mu$, and therefore

$$\langle (t\kappa)^{\mathfrak{B}} \colon \kappa < \alpha\rangle \in \mathsf{C}_\mu\{\langle (t\kappa)^{\mathfrak{B}} \colon \kappa < \alpha\rangle\} \cap V(\mathfrak{B}) \subseteq \mathsf{C}_\mu^{[V(\mathfrak{B})]}\mathsf{R}_y^{\mathfrak{B}} .$$

Thus $\mathsf{R}_x^{\mathfrak{B}} \subseteq \mathsf{C}_\mu^{[V(\mathfrak{B})]}\mathsf{R}_y^{\mathfrak{B}}$. □

§3 Representation using $\mathfrak{Rc}\mathfrak{At}\mathfrak{A}$

The next lemma uses terminology from [HMT71, 2.7.1, 2.7.32]. For every α, NCA_α is the class of algebras which satisfy postulates 1.1.1(C_0)–(C_3), (C_5)–(C_7) of [HMT71]. (See [N86] or [T].)

Lemma 15. *Suppose* $1 \leq \alpha$ *and* $\mathfrak{A}$ *is a normal atomic* α*-dimensional Boolean algebra with operators. Then* $\mathfrak{At}\mathfrak{A}$ *is a suitable structure iff* $\mathfrak{A} \in \mathsf{NCA}_\alpha$.

Proof. Imitate the proof of Theorem 2.7.40, [HMT71], noting that statement (4), p. 456, which corresponds to (C_4), is used only to obtain condition 2.7.40(ii), and conversely. □

Definition 16. For every $\mathfrak{A} \in \mathsf{NCA}_\alpha$, let $\mathsf{R}^{\mathfrak{A}}$ be the mapping from $\mathfrak{A}$ into $\mathfrak{Rc}\mathfrak{At}\mathfrak{A}$ defined by $\mathsf{R}^{\mathfrak{A}}(x) = \bigcup_{x \geq y \in At\mathfrak{A}} \mathsf{R}_y^{\mathfrak{At}\mathfrak{A}}$ for every $x \in A$.

Note that $\mathsf{R}^{\mathfrak{A}}(x) = \mathsf{R}_x^{\mathfrak{At}\mathfrak{A}}$ whenever x is an atom of $\mathfrak{A}$. The next lemma follows immediately from Lemmas 13 and 14.

Lemma 17. *If* $\mathfrak{A} \in \mathsf{NCA}_\alpha$ *and* $\kappa, \lambda, \mu < \alpha$, *then* $\mathsf{R}^{\mathfrak{A}}(\mathsf{d}_{\kappa\lambda}) = \mathsf{D}_{\kappa\lambda}^{[V(\mathfrak{At}\mathfrak{A})]}$ *and* $\mathsf{R}^{\mathfrak{A}}(\mathsf{c}_\mu x) \subseteq \mathsf{C}_\mu^{[V(\mathfrak{At}\mathfrak{A})]}(\mathsf{R}^{\mathfrak{A}}(x))$.

Theorem C. *Suppose* $2 \leq \alpha$, $\mathfrak{A} \in \mathsf{NA}_\alpha$, $\mathfrak{A}$ *is complete, atomic, and satisfies* MGR_n *for* $n = 2, 3$. *Then* $\mathfrak{A} \cong \mathfrak{Rc}\mathfrak{At}\mathfrak{A} \in \mathsf{RICs}_\alpha$.

Proof. By Lemma 15, $\mathfrak{At}\mathfrak{A}$ is a suitable structure. By Lemma 12(ii), $\mathsf{R}^{\mathfrak{A}}$ maps $At\mathfrak{A}$ onto $At\mathfrak{Rc}\mathfrak{At}\mathfrak{A}$, so $\mathsf{R}^{\mathfrak{A}}$ is a Boolean isomorphism since $\mathfrak{A}$ is complete. By Lemma 17, we need only show that if $x, y \in At\mathfrak{A}$ and $\mathsf{R}^{\mathfrak{A}}(x) \subseteq \mathsf{C}_\mu^{[V(\mathfrak{At}\mathfrak{A})]}\mathsf{R}^{\mathfrak{At}\mathfrak{A}}(y)$, then $x \leq \mathsf{c}_\mu y$. To do this we imitate part of the proof of Theorem 1 in [AT].

Whenever $\kappa, \lambda < \alpha$, $\kappa \neq \lambda$, and $x \in A$, let $\mathsf{t}_\kappa^\kappa(x) = x$ and $\mathsf{t}_\lambda^\kappa(x) = \mathsf{d}_{\kappa\lambda} \cdot \mathsf{c}_\kappa x$. Then

(1) $\mathsf{t}_\lambda^\kappa(x) \in At\mathfrak{A}$ whenever $x \in At\mathfrak{A}$,
(2) $\mathsf{t}_\lambda^\kappa(x) = x$ whenever $\mathsf{d}_{\kappa\lambda} \geq x \in At\mathfrak{A}$,
(3) $\mathsf{t}_\lambda^\kappa(x) = \mathsf{t}_\lambda^\kappa(y)$ whenever $x \leq \mathsf{c}_\kappa y$ and $x, y \in At\mathfrak{A}$.

These can be easily proved using [HMT71, 1.2.3, 1.2.5, 1.2.7, and 1.3.9]. The latter results apply to NCA_α, since their proofs rely only on postulates (C_0)–(C_3), (C_7).

Let $\Omega = \bigcup_{n<\omega} {}^{2(n+1)}\alpha$. If $\sigma = \langle \kappa_0, \lambda_0, \ldots, \kappa_n, \lambda_n \rangle \in \Omega$ then let $\sigma^{\mathfrak{A}}(x) = \mathsf{t}_{\lambda_0}^{\kappa_0}\mathsf{t}_{\lambda_1}^{\kappa_1}\ldots\mathsf{t}_{\lambda_n}^{\kappa_n}x$ and let $\sigma^\wedge = [\kappa_0/\lambda_0] | [\kappa_1/\lambda_1] | \ldots | [\kappa_n/\lambda_n]$, where $[\kappa/\lambda]$ is the function mapping α to α which sends κ to λ and leaves every other $\mu < \alpha$ unchanged. Note that for all $\sigma, \tau \in \Omega$, $(\sigma^\frown\tau)^\wedge = \sigma^\wedge | \tau^\wedge$, where $\sigma^\frown\tau$ is the concatenation of σ and τ. By [AT, Lemma 1], we have

(4) if $\sigma, \tau \in \Omega$ and $\sigma^\wedge = \tau^\wedge$, then $\sigma^{\mathfrak{A}}(x) = \tau^{\mathfrak{A}}(x)$ for all $x \in A$.

The merry-go-round identities and (C_4^*) are needed only in the proof of (4).

For every trail $t \in Tr(\mathfrak{At}\mathfrak{A})$, let $[t]$ be the function from α to α such that for every $\kappa < \alpha$, $[t](\kappa)$ is the pointer of $P_7(t\kappa)$. We say that t *collapses* on

$X \subseteq \alpha$ if $|P_7(t\kappa)| = 1$ for every $\kappa \in X$. We prove the next statement by induction on $|t|$.

(5) Assume $t = \langle x_0, \kappa_0, \dots, x_n, \kappa_n\rangle \in Tr(\mathfrak{A}\mathrm{t}\mathfrak{A})$, $\sigma, \tau \in \Omega$, t collapses on $Rg\, \tau^\wedge$, and $\sigma^\wedge|[\langle x_0, \kappa_0\rangle] = \tau^\wedge|[t]$. Then $\sigma^{\mathfrak{A}}(x_0) = \tau^{\mathfrak{A}}(x_n)$.

Suppose first that $|t| = 1$. Then $x_0 = x_n$ and $t = \langle x_0, \kappa_0\rangle$. For any $\kappa < \alpha$, $P_7(t\kappa) = P_7(\langle x_0, k\rangle) = \langle x_0, \lambda\rangle = t\lambda$ for some $\lambda \le \kappa$, *i.e.*, $\lambda = [t](\kappa)$. But $t\lambda$ is reduced, so $P_7(t\lambda) = t\lambda$, hence $\lambda = [t](\lambda)$. This shows that $[t]|[t] = [t]$. To prove (5) in this case, we assume $\sigma, \tau \in \Omega$ and $\sigma^\wedge|[t] = \tau^\wedge|[t]$. (The assumption about t collapsing on $Rg\,\tau^\wedge$ is vacuous, since t collapses on α by virtue of its length.)

If $Id_\alpha = [t]$ then the result follows by (4).

Assume $Id_\alpha \ne [t]$. Since σ and τ are finite sequences, $\sigma^\wedge$ and $\tau^\wedge$ can move only finitely many ordinals. The following set of pairs is therefore finite: $M = \{\langle \sigma^\wedge(\kappa), [t](\sigma^\wedge(\kappa))\rangle : \sigma^\wedge(\kappa) \ne \kappa < \alpha\} \cup \{\langle \tau^\wedge(\kappa), [t](\tau^\wedge(\kappa))\rangle : \tau^\wedge(\kappa) \ne \kappa < \alpha\}$. Since M is finite, there is some $\delta \in \Omega$ which is the result of concatenating all the pairs of M in some order. From $[t]|[t] = [t]$ and the assumption $\sigma^\wedge|[t] = \tau^\wedge|[t]$ it is easy to show that $(\sigma^\frown\delta)^\wedge = (\tau^\frown\delta)^\wedge$, so by (4), $\sigma^{\mathfrak{A}}(\delta^{\mathfrak{A}}(x_0)) = (\sigma^\frown\delta)^{\mathfrak{A}}(x_0) = (\tau^\frown\delta)^{\mathfrak{A}}(x_0) = \tau^{\mathfrak{A}}(\delta^{\mathfrak{A}}(x_0))$. Since $|t| = 1$, if $\langle \kappa, \lambda\rangle \in M$, then $t\kappa\, P_3\, t\lambda$ and $x_0 \le \mathsf{d}_{\kappa\lambda}$. It is therefore easy to show by induction, using (2), that $\delta^{\mathfrak{A}}(x_0) = x_0$. Hence $\sigma^{\mathfrak{A}}(x_0) = \tau^{\mathfrak{A}}(x_0)$, as desired.

For the inductive case, let $n \ge 1$, and assume that (5) holds for all trails of length no more than n. Let $t = \langle x_0, \kappa_0, \dots, x_{n-1}, \kappa_{n-1}, x_n, \kappa_n\rangle$, and set $t' = \langle x_0, \kappa_0, \dots, x_{n-1}, \kappa_{n-1}\rangle$. Therefore $|t| = |t'| + 1 = n + 1$. Assume $\sigma, \tau \in \Omega$, t collapses on $Rg\,\tau^\wedge$, and $\sigma^\wedge|[\langle x_0, \kappa_0\rangle] = \tau^\wedge|[t]$. We wish to prove $\sigma^{\mathfrak{A}}(x_0) = \tau^{\mathfrak{A}}(x_n)$.

Suppose that $t \in Do\,P_1$. Choose $t'' \in Tr(\mathfrak{A}\mathrm{t}\mathfrak{A})$ so that $t\,P_1\,t''$. Obviously $|t''| = |t| - 2 = n - 1 < n$, so we may apply (5) to t''. (This case cannot occur when $n < 3$.) Notice that t'' begins at x_0 and ends at x_n. By the definition of P_1, we clearly have $t\kappa\, P_1\, t''\kappa$ for every $\kappa < \alpha$. Consequently $P_7(t\kappa) = P_7(t''\kappa)$ for all $\kappa < \alpha$. It follows that t'' collapses on $Rg\,\tau^\wedge$ since t is assumed to do so. It also follows that $[t] = [t'']$, and hence $\sigma^\wedge|[\langle x_0, \kappa_0\rangle] = \tau^\wedge|[t] = \tau^\wedge|[t'']$. We may therefore apply the inductive hypothesis to t'', σ and τ, obtaining $\sigma^{\mathfrak{A}}(x_0) = \tau^{\mathfrak{A}}(x_n)$.

This completes the proof of (5) in case $t \in Do\,P_1$. Therefore assume $t \notin Do\,P_1$. We consider two cases.

Suppose there is some $\lambda < \alpha$ such that $\lambda \ne \kappa_{n-1}$ and $x_n \le \mathsf{d}_{\kappa_{n-1}\lambda}$. Set $\upsilon = \tau^\frown\langle \kappa_{n-1}, \lambda\rangle$. Notice that $\upsilon^\wedge = (\tau^\frown\langle \kappa_{n-1}, \lambda\rangle)^\wedge = \tau^\wedge|\langle \kappa_{n-1}, \lambda\rangle^\wedge = \tau^\wedge|[\kappa_{n-1}/\lambda]$, so $Rg\,\upsilon^\wedge \subseteq Rg\,\tau^\wedge \sim\{\kappa_{n-1}\}$.

Next we show t' collapses on $Rg\,\upsilon$. Let $\mu \in Rg\,\upsilon^\wedge$. Then $\mu \in Rg\,\tau^\wedge$, so $|P_7(t\mu))| = 1$ since t collapses on $Rg\,\tau^\wedge$. Also, $\mu \ne \kappa_{n-1}$, so $t\mu\, P_2\, t'\mu$, hence

$P_7(t\mu) = P_7(t'\mu)$, which implies $1 = |P_7(t\mu)| = |P_7(t'\mu)|$. Thus t' collapses on $Rg\,\upsilon$.

Next we show $[t] = [\kappa_{n-1}/\lambda]|[t']$. Since $\lambda \neq \kappa_{n-1}$, we have $t\kappa_{n-1}(P_4 \cup P_4^{-1})t\lambda P_2 t'\lambda$, so $P_7(t\kappa_{n-1}) = P_7(t\lambda) = P_7(t'\lambda)$, hence $[t](\kappa_{n-1}) = [t](\lambda) = [t'](\lambda) = ([\kappa_{n-1}/\lambda]|[t'])(\kappa_{n-1})$. On the other hand, if $\mu \neq \kappa_{n-1}$, then $t\mu P_2 t'\mu$, so $P_7(t\mu) = P_7(t'\mu)$, hence $[t](\mu) = [t'](\mu) = ([\kappa_{n-1}/\lambda]|[t'])(\mu)$. Thus $[t] = [\kappa_{n-1}/\lambda]|[t']$. By assumption, $\sigma^\wedge|[\langle x_0, \kappa_0\rangle] = \tau^\wedge|[t]$, but $\tau^\wedge|[t] = \tau^\wedge|[\kappa_{n-1}/\lambda]|[t'] = \upsilon^\wedge|[t']$, so $\sigma^\wedge|[\langle x_0, \kappa_0\rangle] = \upsilon^\wedge|[t']$. We may apply our inductive hypothesis (5) to t', σ, and υ, obtaining $\sigma^{\mathfrak{A}}(x_0) = \upsilon^{\mathfrak{A}}(x_{n-1})$. However, $\upsilon^{\mathfrak{A}}(x_{n-1}) = (\tau^\frown\langle\kappa_{n-1}, \lambda\rangle)^{\mathfrak{A}}(x_{n-1}) = \tau^{\mathfrak{A}}(\mathsf{t}_\lambda^{\kappa_{n-1}} x_{n-1})$ by the relevant definitions. Furthermore, we have $x_{n-1} \leq \mathsf{c}_{\kappa_{n-1}} x_n$ and $x_n \leq \mathsf{d}_{\kappa_{n-1}\lambda}$, so $\mathsf{t}_\lambda^{\kappa_{n-1}} x_{n-1} = \mathsf{t}_\lambda^{\kappa_{n-1}} x_n = x_n$ by (2) and (3). Therefore, $\sigma^{\mathfrak{A}}(x_0) = \tau^{\mathfrak{A}}(x_n)$. This completes the proof of (5) in the first case.

Suppose there is no $\lambda < \alpha$ such that $\lambda \neq \kappa_{n-1}$ and $x_n \leq \mathsf{d}_{\kappa_{n-1}\lambda}$. We are assuming $t \notin Do\,P_1$, and the assumptions for this case imply that $t\kappa_{n-1} \notin Do\,(P_4 \cup P_4^{-1} \cup P_2)$, so $t\kappa_{n-1}$ is reduced. Therefore $P_7(t\kappa_{n-1}) = t\kappa_{n-1}$. Now $|P_7(t\kappa_{n-1})| = n+1 > 1$ and t collapses on $Rg\,\tau^\wedge$, so $\kappa_{n-1} \notin Rg\,\tau^\wedge$. Choose any $\lambda < \alpha$ such that $\lambda \neq \kappa_{n-1}$. This is possible since $\alpha \geq 2$. Set $\upsilon = \tau^\frown\langle\kappa_{n-1}, \lambda\rangle$. As above, we have $\upsilon^\wedge = \tau^\wedge|[\kappa_{n-1}/\lambda]$, but this time $\kappa_{n-1} \notin Rg\,\tau^\wedge$, so $\upsilon^\wedge = \tau^\wedge$. If $\mu \neq \kappa_{n-1}$ (in particular, if $\mu \in Rg\,\upsilon^\wedge = Rg\,\tau^\wedge$) then $t\mu P_2 t'\mu$, so $P_7(t\mu) = P_7(t'\mu)$ and $[t](\mu) = [t'](\mu) = ([\kappa_{n-1}/\lambda]|[t'])(\mu)$. Now t collapses on $Rg\,\tau^\wedge$, so t' collapses on $Rg\,\upsilon^\wedge$. It is possible that $[t]$ and $[\kappa_{n-1}/\lambda]|[t']$ differ on κ_{n-1}. However, since $\kappa_{n-1} \notin Rg\,\tau^\wedge$ we have $\tau^\wedge = \tau^\wedge|[\kappa_{n-1}/\lambda]$, hence $\tau^\wedge|[t] = \tau^\wedge|[\kappa_{n-1}/\lambda]|[t'] = \upsilon^\wedge|[t']$. By assumption, $\sigma^\wedge|[\langle x_0, \kappa_0\rangle] = \tau^\wedge|[t]$, so $\sigma^\wedge|[\langle x_0, \kappa_0\rangle] = \upsilon^\wedge|[t']$.

We may now apply the inductive hypothesis to t', σ, and τ, obtaining $\sigma^{\mathfrak{A}}(x_0) = \upsilon^{\mathfrak{A}}(x_{n-1})$. Since $x_{n-1} \leq \mathsf{c}_{\kappa_{n-1}} x_n$, we have $\mathsf{t}_\lambda^{\kappa_{n-1}} x_{n-1} = \mathsf{t}_\lambda^{\kappa_{n-1}} x_n$ by (3). Therefore $\sigma^{\mathfrak{A}}(x_0) = \upsilon^{\mathfrak{A}}(x_{n-1}) = \tau^{\mathfrak{A}}(\mathsf{t}_\lambda^{\kappa_{n-1}} x_{n-1}) = \tau^{\mathfrak{A}}(\mathsf{t}_\lambda^{\kappa_{n-1}} x_n) = \upsilon^{\mathfrak{A}}(x_n)$. But we also have $\tau^\wedge|[\langle x_n, \kappa_n\rangle] = \upsilon^\wedge|[\langle x_n, \kappa_n\rangle]$ since $\tau^\wedge = \upsilon^\wedge$, and trails of length 1 collapse on α, so by the inductive hypothesis, applied to $[\langle x_n, \kappa_n\rangle]$, τ, and υ, we get $\tau^{\mathfrak{A}}(x_n) = \upsilon^{\mathfrak{A}}(x_n)$. Hence $\sigma^{\mathfrak{A}}(x_0) = \tau^{\mathfrak{A}}(x_n)$.

This completes the proof of (5).

To finish the proof of the theorem, suppose $x, y \in At\mathfrak{A}$ and $\mathsf{R}^{\mathfrak{A}}(x) \subseteq \mathsf{C}_\mu^{[V(\mathfrak{A}\mathfrak{t}\mathfrak{A})]}\mathsf{R}^{\mathfrak{A}}(y)$. We wish to show $x \leq \mathsf{c}_\mu y$. Since $x \leq \mathsf{c}_\mu x$, we may assume $x \neq y$. Let $s = \langle x, \mu\rangle$. Then $\langle (s\kappa)^{\mathfrak{A}\mathfrak{t}\mathfrak{A}} : \kappa < \alpha\rangle \in \mathsf{R}_x^{\mathfrak{A}}$, so, by our hypothesis, there is some reduced trail $t \in Tr(\mathfrak{A}\mathfrak{t}\mathfrak{A})$ such that $\langle (t\kappa)^{\mathfrak{A}\mathfrak{t}\mathfrak{A}} : \kappa < \alpha\rangle \in \mathsf{R}_y^{\mathfrak{A}}$ and $s\kappa\, Q\, t\kappa$ whenever $\mu \neq \kappa < \alpha$. Choose $\kappa < \alpha$ so that $\mu \neq \kappa$. This is possible since $\alpha \geq 2$. It follows from $s\kappa\, Q\, t\kappa$ that s and t both begin at x. Note that

t ends at y, t collapses on $Rg\,[\mu/\kappa]$ and $[\mu/\kappa]|[s] = [\mu/\kappa]|[t]$. Therefore, by (5), $\mathsf{t}^{\mu}_{\kappa}x = \mathsf{t}^{\mu}_{\kappa}y$, which implies $x \leq c_{\mu}y$. □

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