CANONICAL RELATIVIZED CYLINDRIC SET ALGEBRAS

ROGER D. MADDUX

(Communicated by Andreas Blass)

Abstract. For every suitable relational structure there is a canonical relativized cylindric set algebra. This construction is used to obtain a generalization of Resek's relative representation theorem, and a stronger version of the "Stone type representation theorem" by Andréka and Thompson.

§1. Introduction

Let $2 \leq n < \omega$. $\text{MGR}_n$ is the set of $n$-ary merry-go-round identities:

$$ S_1^\lambda S_2^\kappa_1 S_3^\kappa_2 \ldots S_n^\kappa_{n-1} S_n^\kappa_n \lambda X = S_1^\lambda S_2^\kappa_1 S_3^\kappa_2 \ldots S_{n-1}^\kappa_{n-2} S_n^\kappa_{n-1} \lambda X $$

where $\lambda, \kappa_1, \ldots, \kappa_n$ are distinct ordinals. The merry-go-round identities are defined in D. Resek's dissertation [R75]. (See [R75, pp. 2-3, and 2.3.4, p. 34], or [HMT85, 3.2.88(1)], or [HR75, pp. 382-383]. For all other unexplained terminology and notation see [HMT71] or [HMT85].) One of the most significant results of [R75] is the relative representation theorem:

Theorem A ([R75, 5.23]). Suppose $2 \leq \alpha < \omega$, $\mathfrak{A} \in \text{CA}_\alpha$, $\mathfrak{A}$ is simple, complete, atomic, and satisfies $\text{MGR}_\kappa$ for $2 \leq \kappa < \alpha$. Then $\mathfrak{A}$ is isomorphic to a relativized cylindric set algebra, i.e., $\mathfrak{A} \in \text{IRCS}_\alpha$.

Resek's proof of the relative representation theorem is very long, on the order of 100 typed pages. (The proof shows that $\mathfrak{A}$ must be complete, even though this is not explicitly assumed in the statement of Theorem 5.23 in [R75]. Hence "complete" should be inserted after "atomistic" in the statement of Theorem 4.3 in [HR75].)

Resek's relative representation theorem has the following consequence.

Theorem B ([R75, 5.27]). For every $\alpha \geq 2$ the following are equivalent:

(i) $\mathfrak{A} \in \text{CA}_\alpha \cap \text{IRCS}_\alpha$,

(ii) $\mathfrak{A} \in \text{CA}_\alpha$ and $\mathfrak{A}$ satisfies $\text{MGR}_\kappa$ whenever $2 \leq \kappa < \omega$.

Theorem B shows that $\text{CA}_\alpha \cap \text{IRCS}_\alpha$ is finitely axiomatizable whenever $\alpha$ is finite, and countably schematizable ([HMT85, 4.1.4]) whenever $\alpha$ is infinite.

Received by the editors March 8, 1988.

1980 Mathematics Subject Classification (1985 Revision). Primary 03G15, 20B27.
Resek's proof of Theorem B from Theorem A requires only six pages, but makes use of two unpublished results of L. Henkin.

R. J. Thompson has improved Theorem B in two ways. First, \( \{ \text{MGR}_n : 2 \leq n < \omega \} \) can be replaced by just \( \text{MGR}_2 \) and \( \text{MGR}_3 \). (See [HMT85, 3.2.88]. \( \text{MGR}_2 \) is 3.2.88(2), and \( \text{MGR}_3 \) is 3.2.88(3).) Second, \( \text{CA}_\alpha \) can be replaced by the class \( \text{NA}_\alpha \), whose definition is obtained from that of \( \text{CA}_\alpha \) by replacing postulate \( (C_4) \), \( c_\kappa c_\lambda x = c_\lambda c_\kappa x \), of [HMT71, 1.1.1], by the weaker postulate \( (C_4^*) \), \( c_\lambda c_\kappa x \geq c_\kappa c_\lambda x \cdot d_\mu \) with \( \mu \neq \kappa, \lambda \) (see [AT] or [T]). With these improvements in place, Theorem B is called the Resek-Thompson theorem in [AT]. Thompson's proof of the Resek-Thompson theorem is similar to Resek's and is also quite long. A short proof, due to H. Andréka, is presented in [AT].

Unfortunately, much of the power of the relative representation theorem is lost in the Resek-Thompson theorem. To see this, suppose \( 2 \leq \alpha < \omega \) and \( A \) is a simple complete atomic \( \text{CA}_\beta \) satisfying \( \text{MGR}_\kappa \) for \( 2 \leq \kappa < \alpha \). Then the Resek-Thompson theorem implies that \( A \) is a subalgebra of an algebra in \( \text{IRIC}_\alpha \), but not that \( A \) is itself already in \( \text{IRIC}_\alpha \), as shown by the relative representation theorem. In particular, every simple finite \( \text{CA}_\alpha \) satisfying the merry-go-round identities is isomorphic to a relativized cylindric set algebra, and is not merely embeddable in such an algebra. Clearly what is needed is a short proof of Resek's relative representation theorem with Thompson's improvements. Theorem C below achieves this goal and more: \( \alpha \) can be infinite, \( A \) need not be simple, and the representation is "canonical", in the sense that no arbitrary choices are made in its construction. In particular, the axiom of choice is not used.

**Theorem C.** Suppose \( 2 \leq \alpha \), \( A \in \text{NA}_\alpha \), \( A \) is complete, atomic, and satisfies \( \text{MGR}_\kappa \) for \( \kappa = 2, 3 \). Then \( A \cong \text{Resek} A \in \text{IRIC}_\alpha \).

This theorem implies the Resek-Thompson theorem. For one direction it suffices to note that every algebra in \( \text{ISRIIC}_\alpha \) satisfies \( \text{MGR}_2 \) and \( \text{MGR}_3 \). For the other direction, suppose that \( A \in \text{NA}_\alpha \) and \( A \) satisfies \( \text{MGR}_2 \) and \( \text{MGR}_3 \). Then \( A \) has a complete and atomic extension \( A' \in \text{NA}_\alpha \) which also satisfies \( \text{MGR}_2 \) and \( \text{MGR}_3 \), by [HMT71, 2.7.5, 2.7.13]. By Theorem C, \( A' \in \text{IRIC}_\alpha \), so \( A \in \text{NA}_\alpha \cap \text{ISRIIC}_\alpha \).

The key idea in the proof of Theorem C is the construction of a canonical relativized cylindric set algebra \( \text{Resek} B \) from any suitable relational structure \( B \). (A precise definition of "suitable" is given in Definition 1 below.) For any such structure \( B \), \( \text{Resek} B \) is a complete and atomic relativized cylindric set algebra with unit element \( V(B) \) (Corollary 8) whose atoms are orbits of single sequences under a group of canonical permutations of the base \( U(B) \) (Lemmas 11 and 12). If \( A \) is any atomic algebra which satisfies postulates 1.1.1(C_0) - (C_3), (C_5) - (C_7) of [HMT71], then the atom structure \( \text{At} A \) is suitable. There is a canonical embedding \( R^3 \) of \( A \) into \( \text{Resek} A \) which is a Boolean isomorphism just in case \( A \) is complete. The embedding preserves the diagonal elements and
preserves cylindrification in one direction (Lemma 17). Cylindrification is fully preserved, and the canonical embedding is therefore an isomorphism, just in case \( \mathfrak{A} \) also satisfies \((C_{4}^{*})\), \(\text{MGR}_{2}\), and \(\text{MGR}_{3}\).

Suitable structures arise naturally from certain sets of matrices of atoms of weakly associative relation algebras (defined in [M82]). If \( \mathfrak{A} \) is an atomic weakly associative relation algebra, then \( \mathfrak{A} \) happened to be a suitable structure. (See [M89] for \( \mathfrak{A} \) and \( B_{3} \).) The two-dimensional projection \( R \) of the ternary relation \( V(\mathfrak{A}^{t} \mathfrak{A} B_{3} \mathfrak{A}) \) was used in the first (unpublished) proof of Theorem 5.20 of [M82]. That proof did not appear in [M82] due to the difficulties encountered in trying to formulate a precise mathematical description of \( R \). This paper finally overcomes those difficulties. (Also, the proof which does appear in [M82] yields a useful auxiliary result, namely Theorem 5.19.) I would like to thank Richard L. Kramer for many useful discussions in the early 1980s about \( R \) and how to describe it, and the referee for extensive help in clarifying the arguments and exposition of this paper.

§2. Definition of \( \mathfrak{RcB} \)

**Definition 1.** \( \mathfrak{B} \) is a suitable structure if \( \mathfrak{B} = (B, T_{\kappa}, E_{\kappa \lambda})_{\kappa, \lambda < \alpha} \) where \( \alpha \) is a nonzero ordinal, \( T_{\kappa} \subseteq B \times B \), \( E_{\kappa \lambda} \subseteq B \), and the following conditions hold for all \( \kappa, \lambda, \mu < \alpha \):

1. \( T_{\kappa} \) is an equivalence relation on \( B \),
2. \( E_{\kappa \kappa} = B \),
3. \( E_{\kappa \lambda} = T_{\mu}^{*}(E_{\kappa \mu} \cap E_{\mu \lambda}) \) whenever \( \kappa, \lambda \neq \mu \),
4. \( T_{\kappa} \cap (E_{\kappa \lambda} \times E_{\kappa \lambda}) \subseteq \text{Id} \) whenever \( \kappa \neq \lambda \).

Throughout this section we assume that \( \mathfrak{B} \) is a suitable structure and \( \alpha > 0 \).

**Lemma 2.** For all \( \kappa, \lambda, \mu < \alpha \),

1. \( E_{\kappa \lambda} = T_{\mu}^{*}E_{\kappa \lambda} \) if \( \mu \neq \kappa, \lambda \),
2. \( B = T_{\kappa}^{*}E_{\kappa \lambda} \),
3. \( E_{\kappa \mu} \cap E_{\mu \lambda} \subseteq E_{\kappa \lambda} \),
4. \( E_{\kappa \lambda} = E_{\lambda \kappa} \).

**Proof.** Parts (i)–(iii) are immediate consequences of Definition 1(i)–(iii). For part (iv), we may assume \( \kappa \neq \lambda \). By Definition 1(i)–(iii) we have \( E_{\kappa \lambda} \sim E_{\lambda \kappa} \subseteq T_{\kappa}^{*}(E_{\kappa \lambda} \sim E_{\lambda \kappa}) \cap T_{\kappa}^{*}(E_{\kappa \lambda} \cap E_{\lambda \kappa}) \). But \( T_{\kappa}^{*}(E_{\kappa \lambda} \sim E_{\lambda \kappa}) \cap T_{\kappa}^{*}(E_{\kappa \lambda} \cap E_{\lambda \kappa}) = \emptyset \) by Definition 1(i)(iv), so \( E_{\kappa \lambda} \subseteq E_{\lambda \kappa} \). \( \square \)

**Definition 3.** Let

\[
\text{Tr}(\mathfrak{B}) = \bigcup_{n<\omega} \left\{ (x_{0}, \kappa_{0}, \ldots, x_{n}, \kappa_{n}) \in (n+1)(B \times \alpha) : (\forall i < n)(x_{i} \neq x_{i+1} \land x_{i} T_{\kappa_{i}} x_{i+1}) \right\}.
\]

The sequences in \( \text{Tr}(\mathfrak{B}) \) are called trails of \( \mathfrak{B} \), or \( \mathfrak{B} \)-trails. Let \( t = (x_{0}, \kappa_{0}, \ldots, x_{n}, \kappa_{n}) \) be a trail of \( \mathfrak{B} \). We say that \( t \) begins at \( x_{0} \), \( t \) ends at \( x_{n} \), \( \kappa_{n} \) is
the pointer of $t$, and $t$ has length $|t| = n + 1$. The trail $t$ is reduced if the following conditions hold:

(i) if $1 = |t|$ and $x_0 \in E_{\kappa_0 \lambda}$, then $\kappa_0 \leq \lambda < \alpha$,

(ii) if $1 < |t|$ then $\kappa_{n-1} = \kappa_n$ and for all $\lambda < \alpha$, $x_n \in E_{\kappa_n \lambda}$ iff $\kappa_n = \lambda$,

(iii) if $0 \leq i < |t| - 2$, then either $x_i \neq x_{i+2}$ or $\kappa_i \neq \kappa_{i+1}$.

Finally, for every $\lambda < \alpha$, let $t\lambda = \langle x_0, \kappa_0, \ldots, x_{n-1}, \kappa_{n-1}, x_n, \lambda \rangle$.

All trails have length 1 or more. Condition (iii) applies only to trails of length 3 or more, and states that a reduced trail cannot have a subsequence of the form $\langle x, \lambda, y, \lambda, x \rangle$. If $t$ is a trail, then so is $t\lambda$.

**Definition 4.** Let $Q$ be the smallest equivalence relation on $Tr(\mathcal{B})$ which contains all pairs of $\mathcal{B}$-trails of the following three types:

1. $\langle \langle x_0, \kappa_0, \ldots, x_i, \lambda, y, \lambda, x_i, \kappa_i, \ldots, x_n, \kappa_n \rangle, \langle x_0, \kappa_0, \ldots, x_i, \kappa_i, \ldots, x_n, \kappa_n \rangle \rangle$ where $0 \leq i \leq n$,

2. $\langle \langle x_0, \kappa_0, \ldots, x_n, \lambda, y, \kappa_n \rangle, \langle x_0, \kappa_0, \ldots, x_n, \kappa_n \rangle \rangle$ where $\lambda \neq \kappa_n$,

3. $\langle \langle x_0, \kappa_0, \ldots, x_n, \lambda \rangle, \langle x_0, \kappa_0, \ldots, x_n, \kappa_n \rangle \rangle$ where $x_n \in E_{\lambda \kappa_n}$.

For each $t \in Tr(\mathcal{B})$, let $t^\mathcal{B}$ be the $Q$-class of $t$, i.e., $t^\mathcal{B} = \{ t' : tQt' \}$. Let $U(\mathcal{B}) = \{ t^\mathcal{B} : t \in Tr(\mathcal{B}) \}$. $U(\mathcal{B})$ is called the canonical base for $\mathcal{B}$, or $\mathcal{B}$-base, and the equivalence classes in $U(\mathcal{B})$ are called canonical base points.

The relations used to define $Q$ are “reductions”. A reduction of type (1) consists of the replacement of any subsequence of the form $\langle x, \lambda, y, \lambda, x \rangle$ by $\langle x \rangle$. Equivalent trails have the same beginnings, but may have different ends and different pointers, due to reductions of type (2) and (3).

The intuition behind trails and the definition of $Q$ arises in the following way. Suppose that $\mathcal{B}$ is a suitable structure and $\mathcal{B}$ is already represented by a relativized cylindric set algebra with base $U$. Refer to the elements of $U$ as “base points”. Suppose also that $R_x$ is the set of sequences in $^aU$ associated with each $x \in B$. This representation must have certain properties. First, if $x \in E_{\kappa \lambda}$, then the $\kappa$-term of any sequence associated with $x$ must be the same as the $\lambda$-term of that sequence, or, briefly, $p_x = p_\lambda$ for every $p \in R_x$. Second, if $x T_\kappa y$, then every sequence $p \in R_x$ can have its $\kappa$-term altered to obtain a sequence $p' \in R_y$, hence $p'_\mu = p_\mu$ whenever $\mu \neq \kappa$. We wish to describe the representation $R$ of $\mathcal{B}$ using trails, and then use that description to construct a canonical representation entirely in terms of $\mathcal{B}$-trails.

Let $t = \langle x, \kappa, \ldots \rangle$ be a trail beginning at $x$. Then $t$ may be conceived as instructions which can be applied to any sequence $p \in ^aU$ associated with $x$ (but not applicable to any sequence not associated with $x$). The instructions in $t$ select a particular base point, depending on $p$. It will turn out that $t$ is reduced if there is no shorter trail which leads from $p$ to the base point selected by $t$.

Some examples will show how the instructions should be followed. Let $p \in R_x$. The trail $\langle x, \kappa \rangle$ says, “Select the $\kappa$-term of the sequence.” So, following
\((x, \kappa)\) from \(p\) leads to the base point \(p_\kappa\). The trail \(\langle x, \kappa, y, \lambda \rangle\) says, “Alter the \(\kappa\)-term of the sequence to get a sequence associated with \(y\), and then select the \(\lambda\)-term of the latter sequence.” Following \(\langle x, \kappa, y, \lambda \rangle\) from \(p\) leads to the base point \(p'_\lambda\) for some \(p' \in R_y\) such that \(p_\mu = p'_\mu\) whenever \(\mu \neq \kappa\). Finally, the trail \(\langle x, \kappa, y, \lambda, z, \nu \rangle\) says, “Alter the \(\kappa\)-term of the first sequence to get a sequence associated with \(y\), then alter the \(\lambda\)-term of the latter sequence to get a third sequence associated with \(z\), and then select the \(\nu\)-term of the third sequence.”

These examples show that the pointer has a different role from the other ordinals in a trail. The pointer tells which base point in the final sequence to select, while the other ordinals in the trail tell which term in one intermediate sequence should be altered to get the next sequence.

The instructions may be ambiguous. There may be more than one way to alter a sequence to get a satisfactory new sequence. In could happen, for example, that \(p \in R_x\), \(x T_\kappa y\), \(p', p'' \in R_y\), \(p_\mu = p'_\mu = p''_\mu\) whenever \(\mu \neq \kappa\), and yet \(p' \neq p''\). Following \(\langle x, \kappa, y, \lambda \rangle\) from \(p\) could therefore lead to either \(p'_\lambda\) or \(p''_\lambda\). We could, of course, make the instructions unambiguous by well-ordering the elements of \(U\) and always choosing the first suitable base point as the new \(\kappa\)-term. We could also complicate the structure of trails by adding ordinals to index which base point is being selected at each successive alteration. This procedure would considerably complicate the definition of \(R \in B\). Nevertheless, the main result could be proved using such a construction.

Instead we assume that the representation has a special property which guarantees the instructions in every trail are unambiguous. More precisely, we assume that if \(x, y \in B\), \(p \in R_x\), \(x T_\kappa y\), \(p', p'' \in R_y\), and \(p_\mu = p'_\mu = p''_\mu\) whenever \(\mu \neq \kappa\), then \(p' = p''\). We will derive the definition of \(Q\) under this assumption of unambiguity.

Consider the base points which can be reached from \(p \in R_x\) by following trails which begin at \(x\). Different trails may lead to the same base point. The definition of the equivalence relation \(Q\) is designed so that if two trails are equivalent via \(Q\), then they lead to the same base point. This is why canonical base points are defined as \(Q\)-equivalence classes. Depending on the structure of the given representation, it may or may not be true that trails leading to the same base point are equivalent, but this \(will\) be true in the canonical representation.

To see the need for each of the three types of reductions in \(Q\), consider the following example. Suppose \(p \in R_x\), \(x \neq y\), \(p' \in R_y\), \(x T_0 y\), \(x, y \in E_{23}\), \(p_\mu = p'_\mu\) whenever \(\mu \neq 0\), and \(p_0\), \(p_1\), \(p_2\), \(p'_0\) are distinct. Following \(\langle x, 0 \rangle\) from \(p\) leads to \(p_0\), while following \(\langle x, 0, y, 0 \rangle\) from \(p\) leads to \(p'_0\), a base point distinct from \(p_0\). On the other hand, following either \(\langle x, 1 \rangle\) or \(\langle x, 0, y, 1 \rangle\) from \(p\) leads to the same base point, since \(p_1 = p'_1\) and, by the assumption of unambiguity, there can be no sequence in \(R_y\) other than \(p'\) which coincides with \(p\) at all terms except the 0-term. We must therefore
define \( Q \) so that \( \langle x, 1 \rangle Q \langle x, 0, y, 1 \rangle \). This justifies the inclusion of reductions of type (2) in \( Q \). Following either \( \langle x, 2 \rangle \) or \( \langle x, 3 \rangle \) from \( p \) also leads to the same base point, since \( x \in E_{23} \) and hence \( p_\kappa = p_\lambda \). By including reductions of type (1) in the definition of \( Q \) we accordingly get \( \langle x, 2 \rangle Q \langle x, 3 \rangle \). For any \( \kappa \), following \( \langle x, 0, y, 0, x, \kappa \rangle \) from \( p \) requires first moving from \( p \) to \( p' \) (no other choice is available, by unambiguity), and then from \( p' \) to a sequence in \( R_x \) which differs from \( p' \) only in its 0-term. But there already is such a sequence, namely \( p \). Hence, by unambiguity, the last sequence obtained by following \( \langle x, 0, y, 0, x, \kappa \rangle \) from \( p \) must be \( p \) itself and the selected base point must be \( p_\kappa \). Since this point is also selected by \( \langle x, \kappa \rangle \), we must define \( Q \) so that \( \langle x, 0, y, 0, x, \kappa \rangle Q \langle x, \kappa \rangle \). This is accomplished by including reductions of type (1). It turns out that no other reductions are needed. Even reductions of type (1) are not strictly necessary to construct a relative representation of a suitable structure. By including reductions of type (1) we guarantee that the resulting canonical relative representation is, in fact, unambiguous. Note that the reduced trails in this example are \( \langle x, 0 \rangle, \langle x, 1 \rangle, \langle x, 2 \rangle \), and \( \langle x, 0, y, 0 \rangle \), which lead to \( p_0, p_1, p_2, \) and \( p'_0 \), respectively.

In the next definition we give short names to the reductions and certain other relations. Then in the lemma we show that every canonical base point contains a unique reduced trail, and give an algorithm for computing the unique reduced trail in \( t^{\mathcal{B}} \) from \( t \).

For any binary relation \( X \), the transitive closure of \( X \) is \( X^\omega = X \cup X^2 \cup X^3 \cup \ldots \), where \( X^2 = X \times X \), \( X^3 = X \times X \times X \), etc., and \( Do X \) is the domain of \( X \). For any set \( U \), \( Id_U \) is the identity relation on \( U \).

**Definition 5.** For \( n = 1, 2, 3 \), let \( P_n \) be the set of pairs of \( \mathcal{B} \)-trails of type \( (n) \) in Definition 4. Let \( P_4 = P_3 \cap \{ \langle tK, tl \rangle : \kappa > \lambda \} \), \( P_5 = P_3 \setminus (P_1 \cup P_2) \cap P_3 \), \( P_6 = P_3 \cup P_5 \), \( Z = Tr(\mathcal{B}) ~ Do (P_4 \cup P_5) \), and \( P_7 = P_6^\omega \setminus Id_Z \).

**Lemma 6.**

(i) \( P_2 \) is a function, i.e., \( P_2^{-1} \subseteq Id_{Tr(\mathcal{B})} \),

(ii) \( P_3 \) is an equivalence relation on \( Tr(\mathcal{B}) \),

(iii) \( P_1^{-1} \cap P_1 = Id_{Tr(\mathcal{B})} \cup P_1 \cap P_1^{-1} \),

(iv) \( P_2^{-1} \cap P_1 = P_2 \cup P_1 \setminus P_2^{-1} \) and \( P_1^{-1} \cap P_2 = P_2^{-1} \cup P_2 \setminus P_1^{-1} \),

(v) \( P_1^{-1} \cap P_3 = P_3 \setminus P_1^{-1} \) and \( P_3 \setminus P_1 = P_1 \setminus P_3 \),

(vi) \( P_1^{-1} \cap P_3 \subseteq P_5 \setminus P_5^{-1} \cup P_3 \),

(vii) \( P_2^{-1} \cap P_3 \subseteq P_5 \setminus P_5^{-1} \cup P_5^{-1} \) and \( P_2^{-1} \cap P_1 \subseteq P_5 \setminus P_5^{-1} \cup P_5 \),

(viii) \( P_2^{-1} \cap P_3 \subseteq P_5 \),

(ix) \( P_6^{-1} \cap P_6 \subseteq P_6 \setminus P_6^{-1} \),

(x) \( Q = (P_6 \cup P_6^{-1})^\omega = P_6^\omega \setminus (P_6^{-1})^\omega \),

(xi) for every \( t \in Tr(\mathcal{B}) \), \( t \) is reduced iff \( t \in Z \),

(xii) \( Q = P_6^\omega \setminus Id_Z \ \setminus (P_6^{-1})^\omega = P_7 \setminus P_7^{-1} \),

(xiii) \( Id_Z \setminus P_7 = Id_Z \),

(xiv) \( P_7 \) is a function.
Proof. Parts (i), (iii)–(v) follow just from the relevant parts of Definition 4. Part (vi) follows from Definition 1(ii) and Lemma 2(iii)(iv). Part (vii) follows from (ii), (iv), and (v). Part (viii) follows from Definition 4 and Lemma 2(i). Parts (ii), (vi)–(viii) imply (ix).

We have $\text{Id}_{\text{Tr}(\mathfrak{B})} \subseteq P_1 \cup P_2 \cup P_3 \subseteq P_6 \subseteq Q$ by part (ii), and $Q$ is the equivalence relation generated by $P_1 \cup P_2 \cup P_3$, so $Q = (P_6 \cup P_6^{-1})^\omega$. Part (ix) implies $(P_6 \cup P_6^{-1})^\omega \subseteq (P_6 \cup P_6^{-1})^\omega$ by induction, and the opposite inclusion is trivially true. Thus (x) holds.

If $|t| = 1$ then $t$ is reduced iff $t \notin D_0 P_4$. If $|t| > 1$ then $t$ is reduced iff $t \notin D_0 P_5$. This follows from the observation that if the pointer can be changed by a type (3) reduction, then a type (2) reduction can be performed. More precisely, if $|t| > 1$, $t P_3 t'$, and $t \neq t'$, then $t' = t \lambda$ for some $\lambda$, and hence either $t \in D_0 P_2$ or $t' \in D_0 P_2$. Thus (xi) holds.

Let $R = P_4 \cup P_5$. Then $\text{Tr}(\mathfrak{B}) = Z \cup D_0 R$, so

$$\text{Id}_{\text{Tr}(\mathfrak{B})} = \text{Id}_Z \cup \text{Id}_{D_0 R} \subseteq \text{Id}_Z \cup R^{-1}$$

$$= \text{Id}_Z \cup R\text{Id}_{\text{Tr}(\mathfrak{B})} | R^{-1} \subseteq \text{Id}_Z \cup R(\text{Id}_Z \cup R^{-1}) | R^{-1}$$

$$= \text{Id}_Z \cup R|\text{Id}_Z | R^{-1} \cup R^2 | (R^{-1})^2.$$ Continuing in this way, we get

$$\text{Id}_{\text{Tr}(\mathfrak{B})} = \text{Id}_Z \cup \bigcup_{0 < k < n} R^k | \text{Id}_Z | (R^{-1})^k \cup R^n | (R^{-1})^n$$

whenever $2 \leq n < \omega$. If $t P_3 t'$ then $|t| > |t'|$, and if $t P_4 t \lambda$ then $|t| > \lambda$, so there are no infinite $R$-chains. Therefore, given an arbitrary $t \in \text{Tr}(\mathfrak{B})$, there is some $n$ such that $(t, t) \notin R^n | (R^{-1})^n$ and $2 \leq n < \omega$, which implies, by (1), that $(t, t) \in \text{Id}_Z \cup \bigcup_{0 < k < n} R^k | \text{Id}_Z | (R^{-1})^k$. This proves $\text{Id}_{\text{Tr}(\mathfrak{B})} \subseteq \text{Id}_Z \cup R^\omega | \text{Id}_Z | (R^{-1})^\omega$. Note that $\text{Id}_Z \cup R \subseteq P_6$. Consequently $\text{Id}_{\text{Tr}(\mathfrak{B})} \subseteq P_6 | \text{Id}_Z | (P_6^{-1})^\omega$ and $P_6^\omega | P_6^\omega = P_6^\omega$. By (x), $Q = P_6^\omega | \text{Id}_{\text{Tr}(\mathfrak{B})} | (P_6^{-1})^\omega \subseteq P_6^\omega | \text{Id}_Z | (P_6^{-1})^\omega$. Note that $P_7^{-1} = \text{Id}_Z | (P_6^{-1})^\omega = \text{Id}_Z | (P_6^{-1})^\omega$, so $Q = P_7^{-1}$. Thus (xii) holds.

By the relevant definitions, $\text{Id}_Z | P_6 = \text{Id}_Z | (\text{Id}_{\text{Tr}(\mathfrak{B})} \cup P_4 \cup P_4^{-1} \cup P_5) = \text{Id}_Z \cup \text{Id}_Z | P_4^{-1}$. Also, $P_4^{-1} | P_6 \subseteq P_3 | P_6 = P_6$ by part (ii). Consequently $\text{Id}_Z | P_4^\omega = \text{Id}_Z \cup \text{Id}_Z | P_4^{-1}$ by induction. Also, $P_4^{-1} | \text{Id}_Z = \emptyset$, so $\text{Id}_Z | P_7 = \text{Id}_Z | P_6^\omega | \text{Id}_Z = (\text{Id}_Z \cup \text{Id}_Z | P_4^{-1}) | \text{Id}_Z = \text{Id}_Z$. Thus (xiii) holds.

By the definition of $P_7$, (x), (xii), and (xiii), $P_7^{-1} | P_7 = \text{Id}_Z | (P_6^\omega)^{-1} | P_6^\omega | \text{Id}_Z \subseteq \text{Id}_Z | Q \text{Id}_Z = \text{Id}_Z | P_7 | P_7^{-1} | \text{Id}_Z = \text{Id}_Z$, so (xiv) holds. □

By Lemma 6, $P_7$ is a function contained in $Q$, $P_7(t)$ is reduced, and $t$ is reduced just in case $P_7(t) = t$. Thus $P_7$ maps each trail $t$ to the unique reduced trail $t^{\mathfrak{B}}$. Since $P_3$ is an equivalence relation and $P_3 | P_5 = P_5 = P_5 | P_3$,
we have \( P_6^\omega = (P_3 \cup P_5)^\omega = P_3 \cup P_5^\omega \), so \( P_7 = P_6^\omega\mid \text{Id}_Z = (P_3 \cup P_5^\omega)\mid \text{Id}_Z = (\text{Id}_\text{Tr}(\mathfrak{B}) \cup P_4 \cup P_4^{-1} \cup P_5^\omega)\mid \text{Id}_Z = \text{Id}_Z \cup (P_4 \cup P_5^\omega)\mid \text{Id}_Z \). In other words, to compute \( P_7(t) \) in case \( t \) is not reduced, follow a \( P_5 \)-chain until a trail is obtained which is not in \( \text{Do} P_5 \), and then reduce the pointer as much as possible by using \( P_4 \) once. Notice that \( |P_7(t)| \leq |t| \) for every trail \( t \). If \( |t| = 1 \), then \( t \notin \text{Do} P_5 \), and either \( t \) is already reduced, or else \( t P_4 P_7(t) \), where \( P_7(t) = t\kappa \) for some ordinal \( \kappa \) which is smaller than the pointer of \( t \).

**Definition 7.** For every \( x \in B \), \( R_x^\mathfrak{B} = \{(t\kappa) : \kappa < \alpha : t \in \text{Tr}(\mathfrak{B}), \text{ t ends at } x \} \). Let \( V(\mathfrak{B}) = \bigcup_{x \in B} R_x^\mathfrak{B} \) and \( \mathcal{R}c \mathfrak{B} = \mathcal{R}l_{V(\mathfrak{B})} \mathfrak{A} \), where \( \mathfrak{A} \) is the subalgebra of \( \mathfrak{S}b^\mathfrak{B}U(\mathfrak{B}) \) which is completely generated by \( \{R_x^\mathfrak{B} : x \in B \} \). \( \mathcal{R}c \mathfrak{B} \) is the canonical relativized cylindric set algebra of \( \mathfrak{B} \).

**Corollary 8.** \( \mathcal{R}c \mathfrak{B} \) is a complete atomic relativized cylindric set algebra.

It turns out that \( R_x^\mathfrak{B} \) is an atom in \( \mathcal{R}c \mathfrak{B} \), for every \( x \in B \). To prove this we will use the next lemma.

For every set \( U \), if \( p \) is a one-to-one function from \( U \) into \( U \), then \( \hat{p} \) is the function which maps \( Sb^\mathfrak{A}(U) \) to \( Sb^\mathfrak{A}(U) \) and is defined by \( \hat{p}X = \{a \in U : a[p^{-1}] \in X \} \) for every \( X \subseteq U \). (See [HMT85, p. 15].) It follows that if \( p \) is a permutation of \( U \), then \( \hat{p}X = \{a[p] : a \in X \} \).

**Lemma 9.** Suppose \( U \) and \( B \) are sets, \( R_x \subseteq \mathfrak{A}U \) for every \( x \in B \), and \( P \) is a set of permutations of \( U \). Assume, for every \( x \in B \), that \( P \) preserves \( R_x \), i.e., \( (\forall p \in P) \hat{p}R_x = R_x \), and that \( P \) acts transitively on \( R_x \), i.e., \( (\forall a, b \in R_x) (\exists p \in P) (a[p] = b) \). Let \( \mathfrak{A} \) be the subalgebra of \( \mathfrak{S}b^\mathfrak{A}U \) which is completely generated by \( \{R_x : x \in B \} \). Then \( \{R_x : x \in B\} \subseteq \text{At}\mathfrak{A} \).

**Proof.** \( P \) preserves \( \{R_x : x \in B\} \), and therefore preserves all relations \( S \) in \( \mathfrak{A} \). If some such relation \( S \) were properly contained within some \( R_x \), then it would be possible, by the transitivity of \( P \) on \( R_x \), to pick a \( p \in P \) which moves some sequence from \( S \) to \( R_x \sim S \), contradicting the fact that every \( p \in P \) must preserve \( S \). \( \Box \)

**Definition 10.** For every \( \mathfrak{B} \)-trail \( s = (x_0, \kappa_0, x_1, \kappa_1, \ldots, x_{n-1}, \kappa_{n-1}, x_n, \kappa_n) \) let \( \delta = (x_n, \kappa_{n-1}, x_{n-1}, \kappa_{n-2}, \ldots, x_1, \kappa_0, x_0, \kappa_n) \). If \( t = (y_0, \lambda_0, \ldots, y_m, \lambda_m) \) is any other \( \mathfrak{B} \)-trail, then \( s \circ t \) is defined if \( s \) ends where \( t \) begins, in which case

\[
s \circ t = (x_0, \kappa_0, x_1, \kappa_1, \ldots, x_{n-1}, \kappa_{n-1}, y_0, \lambda_0, \ldots, y_m, \lambda_m).
\]

Also,

\[
L_s(t) = \begin{cases} 
  s \circ t & \text{if } x_n = y_0 \\
  \delta \circ t & \text{if } x_n \neq y_0 = x_0 \\
  t & \text{if } x_n \neq y_0 \neq x_0
\end{cases}
\]

and, for any \( X \subseteq \text{Tr}(\mathfrak{B}) \), \( l_s(X) = \bigcup_{t \in X} L_s(t)^\alpha \). Let

\[
Pm(\mathfrak{B}) = \{l_s : s \in \text{Tr}(\mathfrak{B})\}.
\]
If $s$ and $t$ are $\mathcal{B}$-trails, then $\tilde{s}$ is a $\mathcal{B}$-trail of the same length, and if $s \odot t$ is defined, then $s \odot t$ is also a $\mathcal{B}$-trail with $|s \odot t| = |s| + |t| - 1$. The associative law for $\odot$ holds whenever both sides are defined. If $|s| = 1$, then $\tilde{s} = s$ and $L_s(t) = t$ for every $\mathcal{B}$-trail $t$. Note that $(s \odot t)\kappa = s \odot t \kappa$ and $L_s(t)\kappa = L_s(t \kappa)$.

**Lemma 11.** Suppose $s, t \in Tr(\mathcal{B})$. Then

(i) $l_s(t^\mathcal{B}) = L_s(t)^\mathcal{B}$,

(ii) $t^\mathcal{B} = l_s l_s(t^\mathcal{B}) = l_s l_s(t)^\mathcal{B}$,

(iii) $l_s$ is a permutation of $U(\mathcal{B})$ and $(l_s)^{-1} = l_s$.

**Proof.** Suppose $s$ begins at $x_0$ and ends at $x_n$. If $t$ begins at $y_0$ then $L_s(t)$ begins at $[x_0, x_n](y_0)$, where $[x_0, x_n]$ is the permutation of $\mathcal{B}$ which interchanges $x_0$ and $x_n$ and fixes all other elements of $\mathcal{B}$. Hence, if $L_s(t)$ and $L_s(t')$ have the same beginning, then so do $t$ and $t'$, and the definitions of $L_s(t)$ and $L_s(t')$ fall into the same cases, i.e., $L_s(t) = t$ iff $L_s(t') = t'$, $L_s(t) = s \odot t$ iff $L_s(t') = s \odot t'$, and $L_s(t) = s \odot t$ iff $L_s(t') = s \odot t'$. It follows easily that $L_s$ is one-to-one. Furthermore, $t, L_s L_s(t)$, and $L_s L_s(t)$ have the same beginning.

According to the definitions of $\odot$ and $L_s$, $t$ is always a final segment of $L_s(t)$. Consequently $L_s$ preserves $P_1$, $P_2$, and $P_3$, i.e., $L_s^{-1}|P_1|L_s \subseteq P_1$, $L_s^{-1}|P_2|L_s \subseteq P_2$, and $L_s^{-1}|P_3|L_s \subseteq P_3$. So $L_s$ also preserves $Q$, i.e., $L_s^{-1}|Q|L_s \subseteq Q$.

To prove (i), first suppose $t' \in l_s(t^\mathcal{B})$. Then there is some $t'' \in t^\mathcal{B}$ such that $t' \in L_s(t'')^\mathcal{B}$. Therefore $t'' \odot t$ and $t' \odot L_s(t'')$. Since $L_s$ preserves $Q$, we have $L_s(t'') \odot L_s(t)$, so $t' \odot L_s(t)$, i.e., $t' \in L_s(t^\mathcal{B})$. Thus $l_s(t^\mathcal{B}) \subseteq L_s(t^\mathcal{B})$.

The opposite inclusion holds trivially.

For the proof of (ii), first check that $L_s L_s(t)$ is either $t$, $s \odot s \odot t$, or $s \odot s \odot t$. We will show $L_s L_s(t) \in t^\mathcal{B}$ by considering three cases. Obviously $L_s L_s(t) \in t^\mathcal{B}$ if $L_s L_s(t) = t$. Suppose $L_s L_s(t) = s \odot s \odot t \neq t$. Then $|s| > 1$ and $s \odot s \odot t \neq t$. In fact, $s \odot s \odot t \in P_1$, so $L_s L_s(t) \in t^\mathcal{B}$. Similarly, if $L_s L_s(t) = s \odot s \odot t \neq t$, then $s \odot s \odot t \in P_1$, so $L_s L_s(t) \in t^\mathcal{B}$. From $L_s L_s(t) \in t^\mathcal{B}$ and part (i) we get $t^\mathcal{B} = (L_s L_s(t))^\mathcal{B} = l_s(L_s(t)^\mathcal{B}) = l_s l_s(t^\mathcal{B})$, which shows one equality of (ii). The other equality holds similarly.

Finally, part (iii) follows from parts (i) and (ii). \(\square\)

By Lemma 11, $Pm(\mathcal{B})$ is a group of permutations of the canonical base points. Hence the functions in $Pm(\mathcal{B})$ are called canonical permutations of $U(\mathcal{B})$.

**Lemma 12.** Let $x \in B$. Then

(i) $Pm(\mathcal{B})$ preserves $R^\mathcal{B}_x$, and $Pm(\mathcal{B})$ acts transitively on $R^\mathcal{B}_x$.

(ii) $R^\mathcal{B}_x \in At\mathcal{R}c \mathcal{B}$.
Proof. By Lemma 9, (i) implies (ii). To prove (i), we first show that $\bar{I}_s R^\mathfrak{B}_x \subseteq R^\mathfrak{B}_x$ for every $s \in Pm(\mathfrak{B})$ using Lemma 11(i):

$$\bar{I}_s R^\mathfrak{B}_x = \left\{ a \mid a \in \{(t\kappa)^\mathfrak{B}_x : \kappa < \alpha \} : t \in Tr(\mathfrak{B}) , \ t \text{ ends at } x \right\}$$

$$= \left\{ ((t\kappa)^\mathfrak{B}_x : \kappa < \alpha) \mid t \in Tr(\mathfrak{B}) , \ t \text{ ends at } x \right\}$$

$$= \left\{ (l_s ((t\kappa)^\mathfrak{B}_x) : \kappa < \alpha) : t \in Tr(\mathfrak{B}) , \ t \text{ ends at } x \right\}$$

$$= \left\{ (L_s (t\kappa)^\mathfrak{B}_x : \kappa < \alpha) : t \in Tr(\mathfrak{B}) , \ t \text{ ends at } x \right\}$$

$$= \left\{ ((L_s (t\kappa)^\mathfrak{B}_x : \kappa < \alpha) : t \in Tr(\mathfrak{B}) , \ t \text{ ends at } x \right\}$$

$$\subseteq R^\mathfrak{B}_x .$$

Since $l_s$ and $\bar{l}_s$ are inverses, so are $\bar{I}_s$ and $\bar{I}_s$. Hence $R^\mathfrak{B}_x = \bar{I}_s \bar{I}_s R^\mathfrak{B}_x \subseteq \bar{I}_s R^\mathfrak{B}_x \subseteq R^\mathfrak{B}_x$, so $R^\mathfrak{B}_x$ is preserved by $l_s$.

To show that $Pm(\mathfrak{B})$ acts transitively on $R^\mathfrak{B}_x$, let us assume that $t$ and $t'$ are trails which end at $x$. We wish to find a canonical permutation $l_s$ which maps $(\langle t\kappa \rangle^\mathfrak{B}_x : \kappa < \alpha)$ to $(\langle t'\kappa \rangle^\mathfrak{B}_x : \kappa < \alpha)$. Let $s = t' \circ \bar{l}$. Note that $s$ is defined since $t'$ ends where $\bar{l}$ begins. Let $\kappa < \alpha$. Then $L_s (t\kappa) = t' \circ \bar{l} \circ t \kappa$.

If $|t| = 1$, then $t' \circ \bar{l} \circ t \kappa = t' \kappa$, so $L_s (t\kappa) \in (t'\kappa)^\mathfrak{B}_x$, and if $|t| > 1$, then $t' \circ \bar{l} \circ t \kappa P_1 \pi_{t' \kappa}$, and again $L_s (t\kappa) \in (t'\kappa)^\mathfrak{B}_x$. Hence $(\langle t\kappa \rangle^\mathfrak{B}_x : \kappa < \alpha) | l_s = (l_s (t\kappa)^\mathfrak{B}_x : \kappa < \alpha) = (\langle t'\kappa \rangle^\mathfrak{B}_x : \kappa < \alpha)$. □

By Lemma 12(i), if $a \in R^\mathfrak{B}_x$, then $R^\mathfrak{B}_x = \{ a \mid p : p \in Pm(\mathfrak{B}) \}$, i.e., every $R^\mathfrak{B}_x$ is the orbit of a single sequence under the group $Pm(\mathfrak{B})$.

Lemma 13. For every $x \in B$ and $\kappa, \lambda < \alpha$, $x \in E_{\kappa \lambda}$ iff $R^\mathfrak{B}_x \subseteq D_{\kappa \lambda}^{[V(\mathfrak{B})]}$.

Proof. If $\kappa = \lambda$ then the result holds by Definition 1(ii), so assume $\kappa \neq \lambda$ and $x \in E_{\kappa \lambda}$. Let $(\langle t\mu \rangle^\mathfrak{B}_x : \mu < \alpha) \in R^\mathfrak{B}_x$. Then $t$ ends at $x$ and $t \kappa P_3 t \lambda$, so $(t\kappa)^\mathfrak{B}_x = (t\lambda)^\mathfrak{B}_x$, which implies $(\langle t\mu \rangle^\mathfrak{B}_x : \mu < \alpha) \in D_{\kappa \lambda}^{[V(\mathfrak{B})]}$.

Now assume $R^\mathfrak{B}_x \subseteq D_{\kappa \lambda}^{[V(\mathfrak{B})]}$. Let $t = (x, \kappa)$. Then $(\langle t\mu \rangle^\mathfrak{B}_x : \mu < \alpha) \in R^\mathfrak{B}_x$, so $t Q t \lambda$. But $t$ and $t \lambda$ are not in the domain of $P_3$ since $|t| = |t \lambda| = 1$, so $t P_3 t \lambda$, i.e., $x \in E_{\kappa \lambda}$. □

Lemma 14. If $x, y \in B, \mu < \alpha$, and $x T_\mu y$, then $R^\mathfrak{B}_x \subseteq C_{\mu}^{[V(\mathfrak{B})]}R^\mathfrak{B}_y$.

Proof. If $x = y$, then $R^\mathfrak{B}_x = R^\mathfrak{B}_y \subseteq C_{\mu}^{[V(\mathfrak{B})]}R^\mathfrak{B}_y$, so we may assume $x \neq y$.

Suppose $(\langle t\kappa \rangle^\mathfrak{B}_x : \kappa < \alpha) \in R^\mathfrak{B}_x$. Let $t' = t \circ (x, \mu, y, \mu)$. Note that $(\langle t'\kappa \rangle^\mathfrak{B}_x : \kappa < \alpha) \in R^\mathfrak{B}_y$. If $\kappa \neq \mu$ then $t \kappa = (t \circ (x, \kappa)) Q (t \circ (x, \mu, y, \kappa)) = t' \kappa$. So $(t\kappa)^\mathfrak{B}_x = (t'\kappa)^\mathfrak{B}_x$ whenever $\kappa \neq \mu$, and therefore

$$(\langle t\kappa \rangle^\mathfrak{B}_x : \kappa < \alpha) \subseteq C_{\mu}^{[V(\mathfrak{B})]} \cap V(\mathfrak{B}) \subseteq C_{\mu}^{[V(\mathfrak{B})]}R^\mathfrak{B}_y.$$ 

Thus $R^\mathfrak{B}_x \subseteq C_{\mu}^{[V(\mathfrak{B})]}R^\mathfrak{B}_y$. □
\section{Representation using $\mathcal{RcAtA}$}

The next lemma uses terminology from [HMT71, 2.7.1, 2.7.32]. For every $\alpha$, $\text{NCA}_\alpha$ is the class of algebras which satisfy postulates 1.1.1\text{-(C3)}, (C5)-(C7) of [HMT71]. (See [N86] or [T].)

**Lemma 15.** Suppose $1 \leq \alpha$ and $\mathfrak{A}$ is a normal atomic $\alpha$-dimensional Boolean algebra with operators. Then $\mathfrak{A}$ is a suitable structure iff $\mathfrak{A} \in \text{NCA}_\alpha$.

**Proof.** Imitate the proof of Theorem 2.7.40, [HMT71], noting that statement (4), p. 456, which corresponds to $(C_4)$, is used only to obtain condition 2.7.40(ii), and conversely. \hfill $\square$

**Definition 16.** For every $\mathfrak{A} \in \text{NCA}_\alpha$, let $R^\mathfrak{A}$ be the mapping from $\mathfrak{A}$ into $\mathcal{RcAtA}$ defined by $R^\mathfrak{A}(x) = \bigcup_{y \geq x} R_y^\mathfrak{A}$ for every $x \in A$.

Note that $R^\mathfrak{A}(x) = R_x^\mathfrak{A}$ whenever $x$ is an atom of $\mathfrak{A}$. The next lemma follows immediately from Lemmas 13 and 14.

**Lemma 17.** If $\mathfrak{A} \in \text{NCA}_\alpha$ and $\kappa, \lambda, \mu < \alpha$, then $R^\mathfrak{A}(d_{\kappa\lambda}) = D_{\kappa\lambda}^{[V(\mathfrak{A})]}$ and $R^\mathfrak{A}(c_{\mu}x) \subseteq C_{\mu}^{[V(\mathfrak{A})]}(R^\mathfrak{A}(x))$.

**Theorem C.** Suppose $2 \leq \alpha$, $\mathfrak{A} \in \text{NA}_\alpha$, $\mathfrak{A}$ is complete, atomic, and satisfies $\text{MGR}_n$ for $n = 2, 3$. Then $\mathfrak{A} \cong \mathcal{RcAtA} \subseteq \text{RICS}_\alpha$.

**Proof.** By Lemma 15, $\mathfrak{A}$ is a suitable structure. By Lemma 12(ii), $R^\mathfrak{A}$ maps $At\mathfrak{A}$ onto $\mathcal{RcAtA}$, so $R^\mathfrak{A}$ is a Boolean isomorphism since $\mathfrak{A}$ is complete. By Lemma 17, we need only show that if $x, y \in At\mathfrak{A}$ and $R^\mathfrak{A}(x) \subseteq C_{\mu}^{[V(\mathfrak{A})]}(R^\mathfrak{A}(y))$, then $x \leq c_{\mu}y$. To do this we imitate part of the proof of Theorem 1 in [AT].

Whenever $\kappa, \lambda < \alpha$, $\kappa \neq \lambda$, and $x \in A$, let $t_{\kappa}^x(x) = x$ and $t_{\lambda}^x(x) = d_{\kappa\lambda} \cdot c_{\kappa}x$.

Then

1. $t_{\kappa}^x(x) \in At\mathfrak{A}$ whenever $x \in At\mathfrak{A}$,
2. $t_{\kappa}^x(x) = x$ whenever $d_{\kappa\lambda} \geq x \in At\mathfrak{A}$,
3. $t_{\lambda}^x(x) = t_{\kappa}^y(y)$ whenever $x \leq c_{\kappa}y$ and $x, y \in At\mathfrak{A}$.

These can be easily proved using [HMT71, 1.2.3, 1.2.5, 1.2.7, and 1.3.9]. The latter results apply to $\text{NCA}_\alpha$, since their proofs rely only on postulates $(C_0)$–$(C_3)$, $(C_7)$.

Let $\Omega = \bigcup_{n<\omega}^{2(n+1)}\alpha$. If $\sigma = (\kappa_0, \lambda_0, \ldots, \kappa_n, \lambda_n) \in \Omega$ then let $\sigma^\mathfrak{A}(x) = t_{\kappa_0}^{\kappa_1} \cdots t_{\kappa_n}^{\lambda_n}x$ and let $\sigma^\wedge = [\kappa_0/\lambda_0][\kappa_1/\lambda_1] \cdots [\kappa_n/\lambda_n]$, where $[\kappa/\lambda]$ is the function mapping $\alpha$ to $\alpha$ which sends $\kappa$ to $\lambda$ and leaves every other $\mu < \alpha$ unchanged. Note that for all $\sigma, \tau \in \Omega$, $(\sigma \circ \tau)^\wedge = \sigma^\wedge \circ \tau^\wedge$, where $\sigma \circ \tau$ is the concatenation of $\sigma$ and $\tau$. By [AT, Lemma 1], we have

4. if $\sigma, \tau \in \Omega$ and $\sigma^\wedge = \tau^\wedge$, then $\sigma^\mathfrak{A}(x) = \tau^\mathfrak{A}(x)$ for all $x \in A$.

The merry-go-round identities and $(C_4^*)$ are needed only in the proof of (4).

For every trail $t \in Tr(At\mathfrak{A})$, let $[t]$ be the function from $\alpha$ to $\alpha$ such that for every $\kappa < \alpha$, $[t](\kappa)$ is the pointer of $P_{\tau}(tk)$. We say that $t$ collapses on
$X \subseteq \alpha$ if $|P_{\gamma}(\kappa)| = 1$ for every $\kappa \in X$. We prove the next statement by induction on $|t|$.

(5) Assume $t = (x_0, \kappa_0, \ldots, x_n, \kappa_n) \in \text{Tr}(\alpha \times \alpha)$, $\sigma, \tau \in \Omega$, $t$ collapses on $Rg \tau^\frown$, and $\sigma^\frown[(x_0, \kappa_0)] = \tau^\frown[|t|]$. Then $\sigma^\alpha(x_0) = \tau^\alpha(x_n)$.

Suppose first that $|t| = 1$. Then $x_0 = x_n$ and $t = (x_0, \kappa_0)$. For any $\kappa < \alpha$, $P_{\gamma}(\kappa) = P_{\gamma}(\langle x_0, \kappa \rangle) = \langle x_0, \lambda \rangle = \lambda \iota$ for some $\lambda \leq \kappa$, i.e., $\lambda = [\rho](\kappa)$. But $\lambda \iota$ is reduced, so $P_{\gamma}(\lambda \iota) = \lambda \iota$, hence $\lambda = [\rho](\lambda)$. This shows that $[\rho][|t|] = [|t|]$.

To prove (5) in this case, we assume $\sigma, \tau \in \Omega$ and $\sigma^\frown[|t|] = \tau^\frown[|t|]$. (The assumption about $t$ collapsing on $Rg \tau^\frown$ is vacuous, since $t$ collapses on $\alpha$ by virtue of its length.)

If $Id_\alpha = [t]$ then the result follows by (4).

Assume $Id_\alpha \neq [t]$. Since $\sigma$ and $\tau$ are finite sequences, $\sigma^\frown$ and $\tau^\frown$ can move only finitely many ordinals. The following set of pairs is therefore finite: $M = \{(\sigma^\frown(\kappa), [\sigma^\frown(\kappa)]) : \sigma^\frown(\kappa) \neq \kappa < \alpha\} \cup \{(\tau^\frown(\kappa), [\tau^\frown(\kappa)]) : \tau^\frown(\kappa) \neq \kappa < \alpha\}$. Since $M$ is finite, there is some $\kappa \delta \in \Omega$ which is the result of concatenating all the pairs of $M$ in some order. From $[\rho][|t|] = [|t|]$ and the assumption $\sigma^\frown[|t|] = \tau^\frown[|t|]$ it is easy to show that $(\sigma^\frown \delta)^\frown = (\tau^\frown \delta)^\frown$, so by (4), $\sigma^\alpha(\delta^\alpha(x_0)) = (\sigma^\frown \delta)^\alpha(x_0) = (\tau^\frown \delta)^\alpha(x_0) = \tau^\alpha(\delta^\alpha(x_0))$. Since $|t| = 1$,

If $\langle \kappa, \lambda \rangle \in M$, then $\kappa \tau P_3 \lambda$ and $x_0 \leq d_\kappa \lambda$. It is therefore easy to show by induction, using (2), that $\delta^\alpha(x_0) = x_0$. Hence $\sigma^\alpha(x_0) = \tau^\alpha(x_0)$, as desired.

For the inductive case, let $n \geq 1$, and assume that (5) holds for all trails of length no more than $n$. Let $t = (x_0, \kappa_0, \ldots, x_{n-1}, \kappa_{n-1}, x_n, \kappa_n)$, and set $t' = (x_0, \kappa_0, \ldots, x_{n-1}, \kappa_{n-1})$. Therefore $|t| = |t'| + 1 = n + 1$. Assume $\sigma, \tau \in \Omega$, $t$ collapses on $Rg \tau^\frown$, and $\sigma^\frown[(x_0, \kappa_0)] = \tau^\frown[|t|]$. We wish to prove $\sigma^\alpha(x_0) = \tau^\alpha(x_n)$.

Suppose that $t \in Do P_1$. Choose $t'' \in \text{Tr}(\alpha \times \alpha)$ so that $t P_1 t''$. Obviously $|t''| = |t| - 2 = n - 1 < n$, so we may apply (5) to $t''$. (This case cannot occur when $n < 3$.) Notice that $t''$ begins at $x_0$ and ends at $x_n$. By the definition of $P_1$, we clearly have $t \kappa P_3 t'' \kappa$ for every $\kappa < \alpha$. Consequently $P_{\gamma}(t \kappa) = P_{\gamma}(t'' \kappa)$ for all $\kappa < \alpha$. It follows that $t''$ collapses on $Rg \tau^\frown$ since $t$ is assumed to do so. It also follows that $[t] = [t'']$, and hence $\sigma^\frown[(x_0, \kappa_0)] = \tau^\frown[|t|] = \tau^\frown[|t''|]$.

We may therefore apply the inductive hypothesis to $t''$, $\sigma$ and $\tau$, obtaining $\sigma^\alpha(x_0) = \tau^\alpha(x_n)$.

This completes the proof of (5) in case $t \in Do P_1$. Therefore assume $t \notin Do P_1$. We consider two cases.

Suppose there is some $\lambda < \alpha$ such that $\lambda \neq \kappa_{n-1}$ and $x_n \leq d_{\kappa_{n-1}} \lambda$. Set $\nu = \tau^\frown(\kappa_{n-1}, \lambda)$. Notice that $\nu^\frown = (\tau^\frown(\kappa_{n-1}, \lambda))^\frown = \tau^\frown[(\kappa_{n-1}, \lambda)]^\frown = \tau^\frown[|\kappa_{n-1}/\lambda|]$, so $Rg \nu^\frown \subseteq Rg \tau^\frown \sim \{\kappa_{n-1}\}$.

Next we show $t'$ collapses on $Rg \nu$. Let $\mu \in Rg \nu^\frown$. Then $\mu \in Rg \tau^\frown$, so $|P_{\gamma}(t \mu)| = 1$ since $t$ collapses on $Rg \tau^\frown$. Also, $\mu \neq \kappa_{n-1}$, so $t \mu P_2 t' \mu$, hence
\( P_\gamma(t\mu) = P_\gamma(t'\mu) \), which implies \( 1 = |P_\gamma(t\mu)| = |P_\gamma(t'\mu)| \). Thus \( t' \) collapses on \( Rg_\nu \).

Next we show \([t] = [\kappa_{n-1}/\lambda][t'] \). Since \( \lambda \neq \kappa_{n-1} \), we have \( t\kappa_{n-1} \in (P_4 \cup P_4^{-1})\lambda P_2 t'\lambda \), so \( P_\gamma(t\kappa_{n-1}) = P_\gamma(t'\lambda) \), hence \( [t]\langle\kappa_{n-1} \rangle = [t']\lambda = [(\kappa_{n-1}/\lambda)[t']]\langle\kappa_{n-1} \rangle \). On the other hand, if \( \mu \neq \kappa_{n-1} \), then \( t\mu P_2 t'\mu \), so \( P_\gamma(t\mu) = P_\gamma(t'\mu) \), hence \( [t]\langle\mu \rangle = [t']\langle\mu \rangle = ((\kappa_{n-1}/\lambda)[t'])\langle\mu \rangle \). Thus \([t] = [\kappa_{n-1}/\lambda][t'] \). By assumption, \( \sigma^\wedge|(x_0,\kappa_0) = \tau^\wedge|[t] \), but \( \sigma^\wedge|[t'] = \tau^\wedge|[\kappa_{n-1}/\lambda][t'] = \nu^\wedge|[t'] \), so \( \sigma^\wedge|(x_0,\kappa_0) = \nu^\wedge|[t'] \). We may apply our inductive hypothesis \((5)\) to \( t' \), \( \sigma \), and \( \nu \), obtaining \( \sigma^\wedge(x_0) = \nu^\wedge(x_{n-1}) \). However, \( \nu^\wedge(x_{n-1}) = (\tau_\wedge(\kappa_{n-1}/\lambda))\alpha(x_{n-1}) = \tau^\wedge(t_{\kappa_{n-1}} x_{n-1}) \) by the relevant definitions. Furthermore, we have \( x_{n-1} \leq c_{\kappa_{n-1}} x_n \) and \( x_n \leq d_{\kappa_{n-1}} \), so \( t_{\kappa_{n-1}} x_{n-1} = t_{\kappa_{n-1}} x_n = x_n \) by \((2)\) and \((3)\). Therefore, \( \sigma^\wedge(x_0) = \tau^\wedge(x_n) \). This completes the proof of \((5)\) in the first case.

Suppose there is no \( \lambda < \alpha \) such that \( \lambda \neq \kappa_{n-1} \) and \( x_n \leq d_{\kappa_{n-1}} \). We are assuming \( t \notin Do_1 \), and the assumptions for this case imply that \( t\kappa_{n-1} \notin Do(P_4 \cup P_4^{-1} \cup P_2) \), so \( t\kappa_{n-1} \) is reduced. Therefore \( P_\gamma(t\kappa_{n-1}) = t\kappa_{n-1} \). Now \( |P_\gamma(t\kappa_{n-1})| = n + 1 > 1 \) and \( t \) collapses on \( Rg_\tau^\wedge \), so \( \kappa_{n-1} \notin Rg_\tau^\wedge \). Choose any \( \lambda < \alpha \) such that \( \lambda \neq \kappa_{n-1} \). This is possible since \( \alpha > 2 \). Set \( \nu = \tau^\wedge(\kappa_{n-1}/\lambda) \). As above, we have \( \nu^\wedge = \tau^\wedge[\kappa_{n-1}/\lambda] \), but this time \( \kappa_{n-1} \notin Rg_\tau^\wedge \), so \( \nu^\wedge = \tau^\wedge \). If \( \mu \neq \kappa_{n-1} \) (in particular, if \( \mu \notin Rg_\nu^\wedge = Rg_\tau^\wedge \)) then \( t\mu P_2 t'\mu \), so \( P_\gamma(t\mu) = P_\gamma(t'\mu) \) and \( [t]\langle\mu \rangle = [t']\langle\mu \rangle = ((\kappa_{n-1}/\lambda)[t'])\langle\mu \rangle \). Now \( t \) collapses on \( Rg_\tau^\wedge \), so \( t' \) collapses on \( Rg_\nu^\wedge \). It is possible that \([t] \) and \( [\kappa_{n-1}/\lambda] \) differ on \( \kappa_{n-1} \). However, since \( \kappa_{n-1} \notin Rg_\tau^\wedge \) we have \( \tau^\wedge|[t] = \tau^\wedge[\kappa_{n-1}/\lambda][t'] \), hence \( \tau^\wedge|[t] = \tau^\wedge|[\kappa_{n-1}/\lambda][t'] = \nu^\wedge|[t'] \). By assumption, \( \sigma^\wedge|(x_0,\kappa_0) = \tau^\wedge|[t] \), so \( \sigma^\wedge|(x_0,\kappa_0) = \nu^\wedge|[t'] \).

We may now apply the inductive hypothesis to \( t' \), \( \sigma \), and \( \tau \), obtaining \( \sigma^\wedge(x_0) = \nu^\wedge(x_{n-1}) \). Since \( x_{n-1} \leq c_{\kappa_{n-1}} x_n \), we have \( t_{\lambda_{n-1}} x_n = t_{\lambda_{n-1}} x_n \) by \((3)\). Therefore \( \sigma^\wedge(x_0) = \nu^\wedge(x_{n-1}) = \tau^\wedge(t_{\lambda_{n-1}} x_n) = \tau^\wedge(t_{\lambda_{n-1}} x_n) = \nu^\wedge(x_n) \). But we also have \( \tau^\wedge|(x_n,\kappa_n) = \nu^\wedge|(x_n,\kappa_n) \) since \( \tau^\wedge = \nu^\wedge \), and trails of length 1 collapse on \( \alpha \), so by the inductive hypothesis, applied to \([x_n,\kappa_n] \), \( \tau \), and \( \nu \), we get \( \tau^\wedge(x_n) = \nu^\wedge(x_n) \). Hence \( \sigma^\wedge(x_0) = \tau^\wedge(x_n) \).

This completes the proof of \((5)\).

To finish the proof of the theorem, suppose \( x,y \in At_\lambda \) and \( R^\wedge(\lambda) \subseteq C^V_\mu(\lambda_\wedge)R^\wedge(\lambda) \). We wish to show \( x \leq c_\mu y \). Since \( x \leq c_\mu x \), we may assume \( x \neq y \). Let \( s = (x,\mu) \). Then \( (sk)^\wedge(\lambda_\wedge) : \kappa < \alpha \) \( R^\wedge(\lambda_\wedge) \), so, by our hypothesis, there is some reduced trail \( t \in Tr(\lambda_\wedge) \) such that \( (tk)^\wedge(\lambda_\wedge) : \kappa < \alpha \) \( R^\wedge(\lambda_\wedge) \), and \( sk \leq Q \kappa \) whenever \( \mu \neq \kappa < \alpha \). Choose \( \kappa < \alpha \) so that \( \mu \neq \kappa \). This is possible since \( \alpha > 2 \). It follows from \( sk \leq Q \kappa \) that \( s \) and \( t \) both begin at \( x \). Note that
$t$ ends at $y$, $t$ collapses on $Rg[\mu/\kappa]$ and $[\mu/\kappa][s] = [\mu/\kappa][t]$. Therefore, by (5), $t^\mu_x = t^\mu_y$, which implies $x \leq c_\mu y$. $\Box$

REFERENCES


DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50011