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ON SUPERPOSITION OF FUNCTIONS OF BOUNDED φ-VARIATION

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ABSTRACT. J. Ciemnoczolowski and W. Orlicz in [1] have obtained some results concerning superpositions of functions of bounded φ-variation. In this note we show that the assumption in Theorem 1 of [1] that ψ satisfies Δ_2 condition may be dropped. Moreover, Theorem 2.B of [1] is extended to a stronger version.

A function $\varphi: \langle 0, \infty) \to \langle 0, \infty)$ is called a φ-function if it is continuous, nondecreasing and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ for $u \to \infty$. A φ-function φ satisfies condition Δ_2 for small u if $\limsup \varphi(2u)/\varphi(u) < \infty$ for $u \to 0+$. We denote by X the vector space of real functions defined on a closed interval $\langle a, b\rangle$ which vanish at a. For $x \in X$, we denote by $\operatorname{osc}(x; \langle a, b\rangle)$ the oscillation of x on $\langle a, b\rangle$.

Let φ be a φ-function and A be a subset of real numbers. A finite subset π of A with the natural order we will call a partition of A. In general, we will write a non-empty partition π of A in form of an increasing finite sequence $(t_i)_{i=1}^n$. For a real function x defined on A and for a partition π of A we define

$$\operatorname{var}_\varphi(x; \pi) = \begin{cases} 0 & \text{if } \operatorname{card} \pi \le 1 \\ \sum_{i=1}^{n-1} \varphi(|x(t_{i+1}) - x(t_i)|) & \text{if } \operatorname{card} \pi \ge 2 \end{cases}.$$

The value $\operatorname{var}_\varphi(x; A) = \sup_\pi \operatorname{var}_\varphi(x; \pi)$, where the supremum is taken over all partitions of A, is called a φ-variation of x on A. If the φ-variation of x is finite then we say x is of bounded φ-variation. It is easy to see that if A is a closed interval $\langle a, b\rangle$ then the above definition of φ-variation of x on A is equivalent to the classical one ([4], p. 582), which was used in [1]. The class of all functions $x \in X$ of bounded φ-variation is denoted by $V_\varphi\langle a, b\rangle$.

J. Ciemnoczołowski and W. Orlicz have stated in [1] the following theorem concerning superpositions of functions of bounded φ-variation.

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Theorem ([1], Theorem 1). *Let φ be an arbitrary φ-function, ψ a φ-function satisfying Δ_2 for small u. Let F_n be real functions on $(-\infty, \infty)$, $F_n(0) = 0$, $n = 1, 2, \dots$. Then the following are equivalent:*

(a) $\sup_n \operatorname{var}_\psi(F_n(x); \langle a, b\rangle) < \infty$ *for* $x \in V_\varphi\langle a, b\rangle$;

(b) *for every* $r > 0$ *there exists a constant* $C_r > 0$ *such that the inequality* $\psi(|F_n(u_1) - F_n(u_2)|) \le C_r\varphi(|u_1 - u_2|)$ *holds for* $u_1, u_2 \in \langle -r, r\rangle$, $n = 1, 2, \dots$.

We will show that the assumption ψ satisfies condition Δ_2 for small u may be dropped. Namely, one has

Theorem 1. *Let (F_n) be a sequence of real functions defined on $(-\infty, \infty)$ and $F_n(0) = 0$ for $n = 1, 2, \dots$. For every pair φ, ψ of φ-functions the following statements are equivalent:*

(i) *For every sequence (x_n) of functions of X if* $\sup_n \operatorname{var}_\varphi(x_n; \langle a, b\rangle) < \infty$ *then* $\sup_n \operatorname{var}_\psi(F_n(x_n); \langle a, b\rangle) < \infty$.

(ii) *If $x \in V_\varphi\langle a, b\rangle$ then* $\sup_n \operatorname{var}_\psi(F_n(x); \langle a, b\rangle) < \infty$.

(iii) *For every $r > 0$ there exists a constant $C_r > 0$ such that the inequality* $\psi(|F_n(u_1) - F_n(u_2)|) \le C_r\varphi(|u_1 - u_2|)$ *holds for* $u_1, u_2 \in \langle -r, r\rangle$, $n = 1, 2, \dots$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious.

(iii) $\Rightarrow$ (i). If $\sup_n \operatorname{var}_\varphi(x_n; \langle a, b\rangle)$ is finite then $r = \sup_n \operatorname{osc}(x_n; \langle a, b\rangle)$ is finite and by (iii)

$$\sup_n \operatorname{var}_\psi(F_n(x_n); \langle a, b\rangle) \le C_r \sup_n \operatorname{var}_\varphi(x_n; \langle a, b\rangle) < \infty.$$

(ii) $\Rightarrow$ (iii). Assume (ii) holds. Then for every $r = 1, 2, \dots$ the functions F_n are uniformly bounded in common in $\langle -r, r\rangle$ (see [1], proof of Theorem 1). If (iii) does not hold then there exist an integer $r > 0$, a nondecreasing sequence (n_i) of indices, subintervals $\langle u_i, v_i\rangle$ of $\langle -r, r\rangle$ such that

$$d_i = \frac{\psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|)}{\varphi(v_i - u_i)} \to \infty.$$

Since F_{n_i} are uniformly bounded in common in $\langle -r, r\rangle$, we have $v_i - u_i \to 0$. Without loss of generality we may assume that sequences (u_i) and (v_i) are convergent to a point w of $\langle -r, r\rangle$. Passing, if necessary, to a partial sequence, we state that one of the following cases holds:

(A) $u_i \ge w$ for all i;

(B) $v_i \le w$ for all i;

(C) $u_i \le u_{i+1} < w < v_{i+1} \le v_i$ for all i.

If (A) holds then as in ([1], proof of Theorem 1) we construct a function $x \in V_\varphi\langle a, b\rangle$ such that $\sup_n \operatorname{var}_\psi(F_n(x); \langle a, b\rangle) = \infty$. If (B) holds then, setting $G_n(u) = F_n(-u)$ for $u \in (-\infty, \infty)$ and $n = 1, 2, \dots$, the case (A) holds for G_n. Therefore, there exists a function $x \in V_\varphi\langle a, b\rangle$ such that $\sup_n \operatorname{var}_\psi(F_n(x); \langle a, b\rangle) = \sup_n \operatorname{var}_\psi(G_n(x); \langle a, b\rangle) = \infty$.

If (C) holds then there exists an increasing sequence i_j of positive integers such that $\varphi(v_{i_j} - u_{i_j}) \le 2^{-j-2}$ and $d_{i_j} \ge 2 \cdot 4^j$ for $j = 1, 2, \cdots$. We set

$$s_j = \min\{s \in \{1, 2, \dots\} : (2s - 1)\varphi(v_{i_j} - u_{i_j}) \ge 2^{-j-1}\},$$

$$m_0 = 0, \qquad m_j = \sum_{k=1}^{j} 2s_k \qquad \text{for } j = 1, 2, \dots.$$

Given a decreasing sequence (t_k) of points of $\langle a, b\rangle$ convergent to a with $t_1 = b$ and setting $x(a) = 0$ and for $j = 1, 2, \dots$; $k = m_{j-1} + 1, \dots, m_j$; $t \in (t_{k+1}, t_k\rangle$

$$x(t) = \begin{cases} v_{i_j} & \text{for odd } k \\ u_{i_j} & \text{for even } k \end{cases},$$

we obtain a regulated function $x \in X$.

Now, we shall prove that $x \in V_\varphi\langle a, b\rangle$. By ([3], Lemma 1.1) it is enough to show that $\mathrm{var}_\varphi(x; (a, b\rangle) < \infty$. Observe that for every φ-function χ and every function $F: (-\infty, \infty) \to (-\infty, \infty)$ we have

$$\text{(a) } \mathrm{var}_\chi(F(x); (t_{1+m_j}, t_{1+m_{j-1}}\rangle) = (2s_j - 1)\chi(|F(v_{i_j}) - F(u_{i_j})|) \qquad \text{for } j = 1, 2, \dots,$$

$$\text{(b) } \mathrm{var}_\chi(x; (t_{1+m_k}, b\rangle) = \sum_{j=1}^{k} \mathrm{var}_\chi(x; (t_{1+m_j}, t_{1+m_{j-1}}\rangle) + \sum_{j=1}^{k-1} \chi(v_{i_{j+1}} - u_{i_j}) \qquad \text{for } k = 1, 2, \dots.$$

Thus, for a partition $\pi = (r_i)_{i=1}^n$ of $(a, b\rangle$ if $t_{1+m_k} < r_1$ then

$$\begin{aligned} \mathrm{var}_\varphi(x; \pi) &\le \mathrm{var}_\varphi(x; (t_{1+m_k}, b\rangle) \\ &\le \sum_j (2s_j - 1)\varphi(v_{i_j} - u_{i_j}) + \sum_j \varphi(v_{i_{j+1}} - u_{i_j}) \\ &\le \sum_j 2^{-j} + \sum_j \varphi(v_{i_j} - u_{i_j}) \le \frac{5}{4}. \end{aligned}$$

Therefore $x \in V_\varphi\langle a, b\rangle$.

For $j = 1, 2, \dots$, by (a) we have

$$\begin{aligned} \mathrm{var}_\psi(F_{n_{i_j}}(x); \langle a, b\rangle) &\ge \mathrm{var}_\psi(F_{n_{i_j}}(x); (t_{1+m_j}, t_{1+m_{j-1}}\rangle) \\ &= (2s_j - 1)\psi(|F_{n_{i_j}}(v_{i_j}) - F_{n_{i_j}}(u_{i_j})|) \\ &\ge (2s_j - 1) \cdot 2 \cdot 4^j \varphi(v_{i_j} - u_{i_j}) \ge 2^j. \end{aligned}$$

Thus $\sup_n \mathrm{var}_\psi(F_n(x); \langle a, b\rangle) = \infty$ and we get a contradiction. □

If we omit the assumption that ψ satisfies the Δ_2 condition then all consequences of ([1], Theorem 1) remain true, except Theorem 2.B.

For a φ-function φ and a real function F defined on $(-\infty, \infty)$ we will write $F \in \mathrm{GL}_\varphi$ if $F(0) = 0$ and F satisfies the following generalized Lipschitz condition: for every $k > 0$ there exists a constant $C_k > 0$ such that $\varphi(|F(u) - F(v)|) \leq C_k \varphi(|u-v|)$ for $u, v \in \langle -k, k\rangle$. It is easy to see that functions from GL_φ are continuous. If $\varphi(u) = u$ then we will write GL instead GL_φ. Using this notation the Theorem 2.A of [1] may be written as follows: $F(V_\varphi\langle a, b\rangle) \subset V_\varphi\langle a, b\rangle$ iff $F \in \mathrm{GL}_\varphi$.

J. Ciemnoczołowski and W. Orlicz have formulated in [1] a sufficient condition for the equality $\mathrm{GL}_\varphi = \mathrm{GL}$. Namely, they have proved the following

Theorem ([1], Theorem 2.B). *Let φ be a strictly increasing φ-function such that φ and φ^{-1} satisfy condition Δ_2 for small u. Then $\mathrm{GL}_\varphi = \mathrm{GL}$.*

Theorems 2 and 3 jointly allow to formulate the necessary and sufficient condition for the equality $\mathrm{GL}_\varphi = \mathrm{GL}$.

Theorem 2. *The inclusion $\mathrm{GL}_\varphi \subset \mathrm{GL}$ holds if and only if φ satisfies the condition*

(E) *for every $c > 0$ there exists a number $r > 0$ such that*

$$\limsup_{u \to 0^+} \frac{\varphi(ru)}{\varphi(u)} > c.$$

To prove this theorem we need a simple lemma.

Lemma. *Let F be a real function defined on $\langle a, b\rangle$. Then for every positive integer n there exist points s, t of $\langle a, b\rangle$ such that*

$$t - s = \frac{b-a}{n} \tag{1}$$

and

$$\frac{|F(t) - F(s)|}{t-s} \geq \frac{|F(b) - F(a)|}{b-a}. \tag{2}$$

Proof. If for some integer $n > 0$ and every pair s, t of points of $\langle a, b\rangle$, satisfying (1), the inequality

$$\frac{|F(t) - F(s)|}{t-s} < \frac{|F(b) - F(a)|}{b-a}$$

holds, then setting $s_k = a + (k-1)(b-a)/n$ and $t_k = a + k(b-a)/n$ for $k = 1, \ldots, n$, we have

$$\begin{aligned} |F(b) - F(a)| &\leq \sum_{k=1}^{n} |F(t_k) - F(s_k)| \\ &< \frac{|F(b) - F(a)|}{b-a} \sum_{k=1}^{n} (t_k - s_k) = |F(b) - F(a)| \end{aligned}$$

and we get a contradiction. □

Proof of Theorem 2. First, suppose that φ satisfies the condition (E) and that $F \in \mathrm{GL}_\varphi \setminus \mathrm{GL}$. Let m be a positive number such that for every $k > 0$ there exists a subinterval $\langle u_k, v_k\rangle$ of $\langle -m, m\rangle$ such that $|F(v_k) - F(u_k)| \geq k(v_k - u_k)$. Let $C > 0$ be such that $\varphi(|F(u) - F(v)|) \leq C\varphi(|u - v|)$ for $u, v \in \langle -m, m\rangle$.

For some $r > 0$ we have

$$\limsup_{u\to 0^+} \frac{\varphi(ru)}{\varphi(u)} > C + 1$$

and there exists a subinterval $\langle u, v\rangle$ of $\langle -m, m\rangle$ such that $|F(v) - F(u)| \geq (r+1)(v-u)$. Since F is continuous, it follows that there exists an $\varepsilon \in (0, v-u)$ such that

$$(+) \qquad |F(v) - F(s)| \geq r(v - s) \quad \text{for } s \in \langle u, u + \varepsilon\rangle.$$

Choosing a number $w \in (0, \varepsilon)$ so that

$$(++) \qquad \varphi(rw) > (C + 1)\varphi(w),$$

we have $v - lw \in \langle u, u+\varepsilon\rangle$ for some integer $l > 0$. Thus, by (+) $|F(v) - F(v - lw)| \geq rlw$ and therefore, by our Lemma there exists a subinterval $\langle u', v'\rangle$ of $\langle v - lw, v\rangle$ such that $v' - u' = w$ and $|F(v') - F(u')| \geq r(v' - u')$. Thus,

$$\begin{aligned}\varphi(rw) = \varphi(r(v' - u')) &\leq \varphi(|F(v') - F(u')|)\\ &\leq C\varphi(v' - u') < (C + 1)\varphi(w),\end{aligned}$$

which contradicts (++). So if φ satisfies the condition (E) then $\mathrm{GL}_\varphi \subset \mathrm{GL}$.

Conversely, suppose that φ does not satisfy (E). Then there exists a constant $c > 1$ such that for every $n = 1, 2, \ldots$ there exists a number $v_n > 0$ such that $\varphi((n + 1)u) \leq c\varphi(u)$ for $u \in (0, v_n)$. For a sequence (u_n) of positive numbers such that $u_1 \leq 1$, $u_n < v_n$ and $2(n + 1)u_{n+1} < nu_n$ for $n = 1, 2, \ldots$, the series $\sum_n (-1)^n nu_n$ is convergent. Now, we define a real function on $(-\infty, \infty)$, setting

$$t_n = -\sum_{i=n}^{\infty} u_i \qquad \text{for } n = 1, 2, \ldots;$$

$$F(t) = 0 \text{ for } t \geq 0; \qquad F(t) = \sum_n (-1)^n nu_n \qquad \text{for } t < t_1;$$

$$F(t_n) = \sum_{i=n}^{\infty} (-1)^i iu_i \qquad \text{for } n = 1, 2, \ldots.$$

Finally, we define F to be a linear function on each $\langle t_n, t_{n+1}\rangle$. Observe that for every integer $n > 0$ and every $u \in \langle u_{n+1}, u_n)$ we have $|F(d + u) - F(d)| \leq (n + 1)u_{n+1}$ for $d \geq t_{n+1}$ and $|F(d + u) - F(d)| \leq nu$ for $d < t_{n+1}$. Hence for every real number d

$$(+++) \qquad |F(d + u) - F(d)| \leq (n + 1)u \qquad \text{for } u \subset \langle u_{n+1}, u_n).$$

More, observe that for $u \geq u_1$ and for every real number d

$$(++++) \qquad |F(d+u) - F(d)| \leq u_1.$$

Given two different real numbers w, w', if $|w - w'| \geq u_1$ then by $(++++)$ $\varphi(F(w) - F(w')|) \leq \varphi(u_1) \leq c\varphi(|w - w'|)$. If $|w - w'| \in \langle u_{n+1}, u_n)$ then by $(+++)$

$$\varphi(|F(w) - F(w')|) \leq \varphi((n+1)|w - w'|) \leq c\varphi(|w - w'|),$$

because $u_n < v_n$. We have proved that $F \in \mathrm{GL}_\varphi$. Finally, for $n = 1, 2, \ldots$ we have $t_n \in \langle -2, 2 \rangle$ and

$$\frac{|F(t_{n+1}) - F(t_n)|}{|t_{n+1} - t_n|} = \frac{nu_n}{u_n} = n.$$

Thus $F \notin \mathrm{GL}$. So if $\mathrm{GL}_\varphi \subset \mathrm{GL}$ then φ satisfies the condition (E). □

Theorem 3. *The inclusion* $\mathrm{GL} \subset \mathrm{GL}_\varphi$ *holds if and only if* φ *satisfies the condition* Δ_2 *for small* u.

Proof. First, assume that φ satisfies Δ_2 for small u and $F \in \mathrm{GL}$. Given $m > 0$, there exists a constant $C > 0$ such that $\varphi(|F(u) - F(v)|) \leq \varphi(C|u - v|)$ for $u, v \in \langle -m, m \rangle$. Because φ satisfies Δ_2 for small u, there exists a constant $K_m > 0$ such that $\varphi(Cw) \leq K_m\varphi(w)$ for $0 \leq w \leq 2m$ (cf. [2], 1.02). Thus, $\varphi(|F(u) - F(v)|) \leq K_m\varphi(|u - v|)$ for $u, v \in \langle -m, m \rangle$. It follows that $F \in \mathrm{GL}_\varphi$.

Conversely, suppose that φ does not satisfy the condition Δ_2 for small u. Then

$$\limsup_{u \to 0^+} \frac{\varphi(2u)}{\varphi(u)} = \infty.$$

and it is easy to see that for $F(u) = 2u$ we have $F \in \mathrm{GL}$ and $F \notin \mathrm{GL}_\varphi$. □

The following result is just exactly a generalization of Theorem 2.B of [1].

Corollary. *The identity* $\mathrm{GL} = \mathrm{GL}_\varphi$ *holds if and only if* φ *satisfies the conditions* (E) *and* Δ_2 *for small* u.

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