

CERTAIN POSITIVE-DEFINITE KERNELS

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ABSTRACT. In one way or another, the extension of the standard Brownian motion process $\{B_t: t \in [0, \infty)\}$ to a (Gaussian) random field $\{B_t: t \in \mathbf{R}_+^d\}$ involves a proof of the positive semi-definiteness of the kernel used to generalize $\rho(s, t) = \text{cov}(B_s, B_t) = s \wedge t$ to multidimensional time. Simple direct analytical proofs are provided here for the cases of (i) the Lévy multiparameter Brownian motion, (ii) the Chentsov Brownian sheet, and (iii) the multiparameter fractional Brownian field.

1. INTRODUCTION

The first random field extension of the standard Brownian motion process $\{B_t: t \in [0, \infty)\}$ to "multidimensional time" $\mathbf{x} \in \mathbf{R}^d$ is due to Paul Lévy [3], [4]. As defined by Lévy, the *multiparameter (standard) Brownian motion* is a real-valued Gaussian random field $\{B_{\mathbf{x}}: \mathbf{x} \in \mathbf{R}^d\}$ having mean zero and covariance kernel defined by

$$(1.1) \quad \Gamma(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \{ \|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{x} - \mathbf{y}\| \}, \quad (\mathbf{x}, \mathbf{y} \in \mathbf{R}^d),$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbf{R}^d . The proof of existence of such a random field can proceed from the standard Kolmogorov construction methods once it is shown that the kernel defined by (1.1) is positive semi-definite; i.e., for any $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbf{R}^d$, $k \geq 1, c_1, \dots, c_k \in \mathbf{R}$,

$$(1.2) \quad \sum_{1 \leq i, j \leq k} \Gamma(\mathbf{x}_i, \mathbf{x}_j) c_i c_j \geq 0.$$

The positive semi-definiteness problem (1.2) for the particular kernel defined by (1.1) had already been solved by Schoenberg [13] a few years prior to Lévy's extension of Brownian motion. In fact, Lévy used Schoenberg's solution in his

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construction; see Lévy [5] for this as well as a systematic account of his results on the sample path regularity of $\{B_{\mathbf{x}}\}$.

An alternate extension of Brownian motion was introduced by Chentsov [1]. Chentsov's Brownian motion, $\{W_{\mathbf{x}} : \mathbf{x} \in \mathbf{R}_+^d\}$, is the mean zero Gaussian random field with covariance kernel defined by

$$(1.3) \quad G(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^d x_i \wedge y_i, \quad (\mathbf{x} = (x_1, \dots, x_d), \mathbf{y} = (y_1, \dots, y_d) \in \mathbf{R}_+^d),$$

where $a \wedge b = \min\{a, b\}$, $a, b \in \mathbf{R}$. This process is often referred to as the *Brownian sheet*. Fundamental distinctions between the Lévy and the Chentsov extensions are readily seen by considering the *increments*. Specifically, defining the *partial order* \leq on \mathbf{R}^d by $(x_1, \dots, x_d) = \mathbf{x} \leq \mathbf{y} = (y_1, \dots, y_d)$ if and only if $x_i \leq y_i$ for each $i = 1, \dots, d$, one has independence of $W_{\mathbf{y}} - W_{\mathbf{x}}$ and $W_{\mathbf{x}}$ if and only if $\mathbf{x} \leq \mathbf{y}$. In contrast, $B_{\mathbf{y}} - B_{\mathbf{x}}$ and $B_{\mathbf{x}}$ are *not* independent, unless \mathbf{x} and \mathbf{y} lie along a common ray emanating from the origin. Nonetheless, it is still possible to establish a unifying representation of these two distinct random fields by viewing them as set-indexed processes; see Ossiander and Pyke [10].

A third family of Gaussian random fields arises quite naturally in considerations of extensions of the *scaling* (or *self-similarity*) properties of the Brownian field; see Mandelbrot [7]. Namely, to extend the properties $\{B_{\lambda\mathbf{x}}\} \stackrel{d}{=} \{\lambda^{1/2} B_{\mathbf{x}}\}$, $(\lambda > 0)$, and/or $\{W_{\lambda\mathbf{x}}\} \stackrel{d}{=} \{\lambda^{d/2} W_{\mathbf{x}}\}$, $(\lambda > 0)$, where $\stackrel{d}{=}$ denotes equality in distribution, let $\{Z_{\mathbf{x}}\}$ be the Gaussian random field having zero mean and (for fixed β) covariance kernel defined by

$$(1.4) \quad \Gamma_{\beta}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \{ \|\mathbf{x}\|^{2\beta} + \|\mathbf{y}\|^{2\beta} - \|\mathbf{x} - \mathbf{y}\|^{2\beta} \}.$$

Then, provided existence can be established, one has $\{Z_{\lambda\mathbf{x}}\} \stackrel{d}{=} \{\lambda^{\beta} Z_{\mathbf{x}}\}$, $(\lambda > 0)$. The Gaussian random field $\{Z_{\mathbf{x}}\}$ with kernel (1.4) is then referred to as the *fractional Brownian field* with *scaling exponent* β . Schoenberg's method for the case $\beta = \frac{1}{2}$ does *not* extend to the positive semi-definiteness problem with other values of β . A proof that $\Gamma_{\beta}(\mathbf{x}, \mathbf{y})$ is positive semi-definite for $0 \leq \beta \leq 1$ first appeared as an application of very general results of R. Gangolli [2] on the positive semi-definiteness of kernels of the so-called *Lévy-Schoenberg type*. In fact the special class of kernels (1.4) provided much of the motivation for Gangolli's general theory. Other proofs are obtained as applications of (relatively deep) white noise integral representations; see for example Ossiander [9], Wong and Zakai [14], Mandelbrot and Van Ness [8] and Pflug [11].

The present paper grew out of our desire to have simple direct solutions of the positive semi-definiteness problem for these kernels. As noted by Gangolli [2], while it is possible that such proofs may be known for these specific kernels, they do not seem to be available in the literature. The solution presented here is obtained by an analysis of a formula suggested by a Poisson approximation proposed by Mandelbrot [6]. This solution is in fact accessible to analysts

with no familiarity with the probabilistic motivation or applications for this problem.

2. THE FRACTIONAL BROWNIAN KERNEL

Lévy’s kernel (1.1) is a special case of the fractional kernel (1.4) with $\beta = \frac{1}{2}$. The proof of (1.4) will be established for all $0 < \beta < 1$. Note that the cases $\beta = 0$ and $\beta = 1$ may be checked directly as

$$(2.1) \quad \sum_{1 \leq i, j \leq k} \Gamma_0(\mathbf{x}_i, \mathbf{x}_j) c_i c_j = \sum_{1 \leq i, j \leq k} \frac{1}{2} c_i c_j = \frac{1}{2} \left(\sum_{i=1}^k c_i \right)^2 \geq 0,$$

and

$$(2.2) \quad \begin{aligned} \sum_{1 \leq i, j \leq k} \Gamma_1(\mathbf{x}_i, \mathbf{x}_j) c_i c_j &= \sum_{1 \leq i, j \leq k} \langle \mathbf{x}_i, \mathbf{x}_j \rangle c_i c_j = \left\langle \sum_{i=1}^k \mathbf{x}_i c_i, \sum_{j=1}^k \mathbf{x}_j c_j \right\rangle, \\ &= \left\| \sum_{i=1}^k \mathbf{x}_i c_i \right\|^2 \geq 0. \end{aligned}$$

For fixed $0 < \beta < 1$, let

$$(2.3) \quad \gamma_\beta(s, r) = |s - r|^{\beta - \frac{1}{2}} \text{sign}(s - r) + |r|^{\beta - \frac{1}{2}} \text{sign}(r), \quad s, r \in \mathbf{R}^1.$$

Lemma 2.1. *Let $0 < \beta < 1$, $-\infty < s, t < \infty$. Then $\gamma_\beta(t, \cdot) \in L_2(\mathbf{R}^1, dr)$ for each t , and*

$$(2.4) \quad \langle \gamma_\beta(t, \cdot), \gamma_\beta(s, \cdot) \rangle = c_\beta (|t|^{2\beta} + |s|^{2\beta} - |t - s|^{2\beta})$$

where c_β is a positive constant and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L_2(\mathbf{R}^1, dr)$ defined by

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(r)g(r) dr, \quad f, g \in L_2(\mathbf{R}^1, dr).$$

Proof. First note that $\gamma_\beta(0, r) = 0$ for all r , and for $t \neq 0$.

$$(2.5) \quad \gamma_\beta^2(r, t) = |t|^{2\beta - 1} \cdot \gamma_\beta^2(1, \frac{t}{t}),$$

Now, both $|1 - u|^{\beta - \frac{1}{2}} \text{sign}(u)$ and $|u|^{\beta - \frac{1}{2}} \text{sign}(u)$ are locally square integrable in neighborhoods of $u = 0$ and $u = 1$, so $\gamma_\beta^2(1, u)$ is as well. Moreover, for $\beta < 1$,

$$\begin{aligned} &\int_{-\infty}^{-2} \gamma_\beta^2(1, u) du + \int_2^{\infty} \gamma_\beta^2(1, u) du \\ &= 2 \int_2^{\infty} \gamma_\beta^2(1, u) du = 2 \int_2^{\infty} [u^{\beta - \frac{1}{2}} - (u - 1)^{\beta - \frac{1}{2}}]^2 du \\ &= 2 \int_2^{\infty} \left(\int_{u-1}^u \left(\beta - \frac{1}{2} \right) v^{\beta - \frac{3}{2}} dv \right)^2 du \leq 2 \left(\beta - \frac{1}{2} \right)^2 \int_2^{\infty} \int_{u-1}^u v^{2\beta - 3} dv du \\ &\leq 2 \left(\beta - \frac{1}{2} \right)^2 \int_2^{\infty} (u - 1)^{2\beta - 3} du < \infty. \end{aligned}$$

Thus, $\gamma_\beta(t, \cdot) \in L_2(\mathbf{R}^1, dr)$ for each t and by (2.5) we have $\|\gamma_\beta(t, \cdot)\|_2^2 = 2c_\beta |t|^{2\beta}$ where $2c_\beta := \|\gamma_\beta(1, \cdot)\|_2^2 > 0$. By linearity of $L_2(\mathbf{R}^1, dr)$, it also follows that $\gamma_\beta(t, \cdot) - \gamma_\beta(s, \cdot) \in L_2(\mathbf{R}^1, dr)$. Moreover

$$\begin{aligned} \|\gamma_\beta(t, \cdot) - \gamma_\beta(s, \cdot)\|_2^2 &= \int_{-\infty}^{\infty} \{|t-r|^{\beta-\frac{1}{2}} \text{sign}(t-r) - |s-r|^{\beta-\frac{1}{2}} \text{sign}(s-r)\}^2 dr \\ &= \int_{-\infty}^{\infty} \{|t-s+x|^{\beta-\frac{1}{2}} \text{sign}(t-s+x) - |x|^{\beta-\frac{1}{2}} \text{sign}(x)\}^2 dx \\ &= \int_{-\infty}^{\infty} \{|t-s-y|^{\beta-\frac{1}{2}} \text{sign}(t-s-y) + |y|^{\beta-\frac{1}{2}} \text{sign}(y)\}^2 dy \\ &= \|\gamma_\beta(t-s, \cdot)\|_2^2 = 2c_\beta |t-s|^{2\beta}. \end{aligned}$$

Thus, the result follows from bilinearity of the inner product as expressed in the simple identity

$$\langle \gamma_\beta(t, \cdot), \gamma_\beta(s, \cdot) \rangle = \frac{1}{2} \{ \|\gamma_\beta(t, \cdot)\|_2^2 + \|\gamma_\beta(s, \cdot)\|_2^2 - \|\gamma_\beta(t, \cdot) - \gamma_\beta(s, \cdot)\|_2^2 \}. \quad \square$$

Theorem 2.1. *The kernel $\Gamma_\beta(\mathbf{x}, \mathbf{y})$ is positive semi-definite for each $0 < \beta < 1$.*

Proof. For each $d \geq 1$, $\mathbf{x}, \mathbf{y} \in \mathbf{R}^d$, $\boldsymbol{\theta} \in S^{d-1} := \{\boldsymbol{\theta} \in \mathbf{R}^d : \|\boldsymbol{\theta}\| = 1\}$ we have, by (2.4),

$$\begin{aligned} &\int_{S^{d-1}} \int_{-\infty}^{\infty} \gamma_\beta(\langle \mathbf{x}, \boldsymbol{\theta} \rangle, r) \gamma_\beta(\langle \mathbf{y}, \boldsymbol{\theta} \rangle, r) dr d\boldsymbol{\theta} \\ &= \int_{S^{d-1}} \frac{1}{2} c_\beta \{ |\langle \mathbf{x}, \boldsymbol{\theta} \rangle|^{2\beta} + |\langle \mathbf{y}, \boldsymbol{\theta} \rangle|^{2\beta} - |\langle \mathbf{x} - \mathbf{y}, \boldsymbol{\theta} \rangle|^{2\beta} \} d\boldsymbol{\theta} \\ &= C_{d,\beta} \{ \|\mathbf{x}\|^{2\beta} + \|\mathbf{y}\|^{2\beta} - \|\mathbf{x} - \mathbf{y}\|^{2\beta} \}, \end{aligned}$$

where

$$C_{d,\beta} = \frac{1}{2} c_\beta \int_{S^{d-1}} |\langle \boldsymbol{\varphi}, \boldsymbol{\theta} \rangle|^{2\beta} d\boldsymbol{\theta}$$

does not depend on $\boldsymbol{\varphi} \in S^{d-1}$. Here $\langle \mathbf{x}, \boldsymbol{\theta} \rangle = \sum_{i=1}^d x_i \theta_i$ is the ordinary dot product and $d\boldsymbol{\theta}$ is normalized surface area (Haar measure) on S^{d-1} . Now simply note that for any $\mathbf{x}_1, \dots, \mathbf{x}_\kappa \in \mathbf{R}^d$, $c_1, \dots, c_\kappa \in \mathbf{R}$,

$$\begin{aligned} \sum_{1 \leq i, j \leq \kappa} \Gamma_\beta(\mathbf{x}_i, \mathbf{x}_j) c_i c_j &= \sum_{1 \leq i, j \leq \kappa} c_i c_j \int_{S^{d-1}} \int_{-\infty}^{\infty} \gamma_\beta(\langle \mathbf{x}_i, \boldsymbol{\theta} \rangle, r) \gamma_\beta(\langle \mathbf{x}_j, \boldsymbol{\theta} \rangle, r) dr d\boldsymbol{\theta} \\ &= \int_{S^{d-1}} \int_{-\infty}^{\infty} \left(\sum_{i=1}^{\kappa} c_i \gamma_\beta(\langle \mathbf{x}_i, \boldsymbol{\theta} \rangle, r) \right)^2 dr d\boldsymbol{\theta} \geq 0. \quad \square \end{aligned}$$

3. THE BROWNIAN SHEET

Our approach to the proof of the positive semi-definiteness for the kernel (1.3) is to let $\chi = \max\{x_i^{(\nu)} : i = 1, \dots, k, \nu = 1, \dots, d\}$ for $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbf{R}_+^d$

and consider the compact operator T on $L_2[0, \chi]$ defined by

$$(3.1) \quad Tf(x) = \int_0^\chi f(y)(x \wedge y) dy, \quad (0 \leq x \leq \chi).$$

Since $x \wedge y = y \wedge x$, T is clearly self-adjoint. It follows from the Hilbert-Schmidt Theorem [12, Thm. VI.16] that there is a complete orthonormal basis $\{\varphi_n\}$ for $L_2[0, \chi]$ consisting of eigenfunctions of T with corresponding positive eigenvalues $\{\lambda_n\}$. Thus,

$$(3.2) \quad x \wedge y = \sum_{n=1}^\infty \lambda_n \varphi_n(x) \varphi_n(y), \quad 0 \leq x, y \leq \chi.$$

Theorem 3.1. *The kernel $G(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^d x_i \wedge y_i$, $\mathbf{x}, \mathbf{y} \in \mathbf{R}_+^d$ is positive semi-definite.*

Proof. For $\mathbf{x}_1, \dots, \mathbf{x}_\kappa \in \mathbf{R}_+^d$, $c_1, \dots, c_\kappa \in \mathbf{R}$ we obtain from an eigenfunction expansion of the form (3.2) that

$$\begin{aligned} & \sum_{1 \leq i, j \leq \kappa} G(\mathbf{x}_i, \mathbf{x}_j) c_i c_j \\ &= \sum_{1 \leq i, j \leq \kappa} c_i c_j \prod_{\nu=1}^d \sum_{n_\nu=1}^\infty \lambda_{n_\nu} \varphi_{n_\nu}(x_i^{(\nu)}) \varphi_{n_\nu}(x_j^{(\nu)}) \\ &= \sum_{n_1=1}^\infty \dots \sum_{n_\kappa=1}^\infty \lambda_{n_1} \dots \lambda_{n_\kappa} \sum_{1 \leq i, j \leq \kappa} c_i c_j \varphi_{n_1}(x_i^{(1)}) \dots \varphi_{n_d}(x_i^{(d)}) \\ & \quad \times \varphi_{n_1}(x_j^{(1)}) \dots \varphi_{n_d}(x_j^{(d)}) \\ &= \sum_{n_1=1}^\infty \dots \sum_{n_\kappa=1}^\infty \lambda_{n_1} \dots \lambda_{n_\kappa} \left(\sum_{i=1}^\kappa c_i \varphi_{n_1}(x_i^{(1)}) \dots \varphi_{n_d}(x_i^{(d)}) \right)^2 \geq 0 \end{aligned}$$

since $\lambda_n > 0$ for all $n = 1, 2, \dots$. \square

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