UNITARY REPRESENTATIONS OF LIE GROUPS AND GÅRDING'S INEQUALITY

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Abstract. We prove two versions of Gårding's inequality for strongly elliptic operators in the enveloping Lie algebra associated with a unitary representation of a Lie group. We then deduce a characterization of the differential structure of the representation in terms of the elliptic operators.

1. Introduction

Let $(\mathcal{H}, G, U)$ denote a continuous representation of the connected Lie group $G$ by unitary operators $U(g), g \in G$, on the Hilbert space $\mathcal{H}$. Fix a basis $a_1, \ldots, a_d$ of the Lie algebra $\mathfrak{g}$ of $G$ and let $A_1, \ldots, A_d$ denote the skew self-adjoint generators of the one-parameter subgroups $t \in \mathbb{R} \mapsto U(e^{-t a_i})$. If $\alpha = (\alpha_1, \ldots, \alpha_d), \alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$, we define $A^\alpha = A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_d^{\alpha_d}$ and set $\mathcal{H}_\alpha = \bigcap_\beta D(A^\beta)$. It follows by standard reasoning that $\mathcal{H}_\infty$ is norm dense in $\mathcal{H}$.

A form

$$\xi \in \mathbb{R}^d \rightarrow C_m(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha \in \mathbb{C},$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_d^{\alpha_d}$, is defined to be strongly elliptic if $\text{Re}((-1)^{m/2} P_m(\xi)) > 0$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$, where $P_m$ denotes the principal part of $C_m$, i.e.,

$$P_m = \sum_{|\alpha| = m} c_\alpha \xi^\alpha.$$

Equivalently, $C_m$ is strongly elliptic if there is a $p > 0$ such that

$$\text{Re}((-1)^{m/2} P_m(\xi)) \geq p |\xi|^m$$

for all $\xi \in \mathbb{R}^d$. The largest value $p_m = p_m(c)$ of $p$ for which this estimate is valid is called the ellipticity constant of $C_m$. Note that the strong ellipticity condition implies automatically that $m$ is even.

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Next we associate with each unitary representation \((\mathcal{H}, G, U)\), each basis \(a_1, \ldots, a_d\), of \(g\), and each strongly elliptic form \(C_m\), an operator
\[
A_m(c) = \sum_{\alpha:|\alpha| \leq m} c_{\alpha} A^\alpha
\]
with domain \(D(A_m) = \mathcal{H}_\infty\). We refer to the \(A_m\) as strongly elliptic operators. The simplest example is the Laplacian
\[
\Delta = -\sum_{i=1}^{d} A_i^2
\]
corresponding to the form \(\xi \mapsto -|\xi|^2\).

**Theorem 1.1.** Let \(A_m(c)\) be strongly elliptic. For each \(p \in \langle 0, p_m(c) \rangle\) there is a \(q \geq 0\) such that
\[
\text{Re}(x, A_m(c)x) \geq p(x, \Delta^{m/2}x) - q(x, x)
\]
for all \(x \in \mathcal{H}_\infty\). Moreover, \(q\) can be chosen independently of the particular unitary representation.

An alternative, weaker version of the theorem can be stated in terms of the \(C^n\)-norms of the representation. These are defined by \(\|x\|_0 = \|x\|\) and
\[
\|x\|_n = \sup_{0 \leq r \leq d} \|A_i x\|_{n-1},
\]
where \(A_0 = I\). The \(C^n\)-subspace
\[
\mathcal{H}_n = \bigcap_{\alpha:|\alpha| \leq n} D(A^n)
\]
is a Banach space with respect to \(\|\cdot\|_n\), and \(\mathcal{H}_\infty\) is \(\|\cdot\|_n\)-dense in \(\mathcal{H}_n\) (see, for example, [G]).

**Theorem 1.2.** Let \(A_m(c)\) be strongly elliptic. For each \(p' \in \langle 0, p_m(c) \rangle\) there is a \(q' \geq 0\) such that
\[
\text{Re}(x, A_m(c)x) \geq p'\|x\|^{2m/2} - q'\|x\|^2
\]
for all \(x \in \mathcal{H}_\infty\). Moreover, \(q\) can be chosen independently of the particular unitary representation.

It was established by R. Goodman [G] that \(\mathcal{H}_n = D(\Delta^{n/2})\) and the norm \(\|\cdot\|_n\) on \(\mathcal{H}_n\) is equivalent to the graph norm \(x \mapsto \|\Delta^{n/2}x\| + \|x\|\). But more recently ([R1], [R2]) the following more precise estimates have been obtained; for each \(n = 1, 2, \ldots\) and \(\epsilon > 0\) there is a \(c_n(\epsilon)\) such that
\[
\|x\|_n \leq (1 + \epsilon)\|\Delta^{n/2}x\| + c_n(\epsilon)\|x\|
\]
for all \(x \in \mathcal{H}_n\). Therefore Theorem 1.2 follows easily from Theorem 1.1. Next we sketch the proof of the latter result. It is essentially a consequence of the work of Langlands [L1], [L2].
2. Outline of the proof of Theorem 1.1

If $C_m$ is a strongly elliptic form with ellipticity constant $p_m$ and $p \in (0, p_m)$ then the form $C'_m$ defined by

$$C'_m(\xi) = C_m(\xi) - p(\xi^2)^{m/2}$$

is also strongly elliptic, with ellipticity constant $p'_m = p_m - p$. Since $A_m(c') = A_m(c) - pA^m/2$ the inequality (1.1) is equivalent to the lower semiboundedness property

$$\Re(x, A_m(c')x) \geq -q(x, x).$$

Therefore to prove Theorem 1.1 it suffices to prove that the real part of every strongly elliptic operator $A_m(c)$ is lower semibounded.

Let $A^\dagger_m$ denote the formal adjoint of $A_m$ on $\mathcal{H}_\infty$, i.e.,

$$A^\dagger_m = \sum_{\alpha, |n| \leq m} \bar{c}_\alpha (-1)^{|n|} A^{\alpha*},$$

on $\mathcal{H}_\infty$ where $A^{\alpha*} = A_\alpha^* \cdots A_1^*$. Then $R_m = (A_m + A^\dagger_m)/2$ is a symmetric operator on $\mathcal{H}_\infty$. But it follows from the structure relations of $g$ that $A^{\alpha*} = A^{\alpha}$ modulo lower order terms, i.e.,

$$A^{\alpha*} = A^{\alpha} + \sum_{\beta : |\beta| < |\alpha|} c_{\alpha, \beta} A^\beta,$$

where the $c_{\alpha, \beta}$ are polynomials in the structure constants. Therefore $R_m$ is a strongly elliptic operator associated with a form $C'_m$ whose principal part $P'_m$ is given by

$$P'_m(\xi) = \sum_{\alpha, |n| = m} (\Re c_\alpha)\xi^n.$$

Now it follows from Langlands' [L1] first theorem that $R_m$ is essentially self-adjoint, and from his second theorem that the self-adjoint closure $\overline{R}_m$ generates a continuous semigroup, which is automatically self-adjoint. But then $\overline{R}_m$ is lower semibounded by spectral theory. Hence there is a $q \geq 0$ such that

$$\Re(x, A_m x) = (x, R_m x) \geq -q(x, x)$$

for all $x \in \mathcal{H}_\infty$. Therefore (1.1) follows from the previous reasoning.

Finally, Langlands' [L1] third theorem establishes that the semigroup $S$ generated by $\overline{R}_m$ has a representation independent kernel, i.e.,

$$S_t = \int_G dg p_t(g) U(g),$$

where $dg$ denotes the left invariant Haar measure and $p_t \in L_1(G; dg)$. Since

$$e^{qt} \leq \int_G dg |p_t(g)|$$
one can then choose \( q \) to be independent of the particular unitary representation. This completes the outline of the proof.

3. Differential structure

Let \( A_m = A_m(c) \) be a strongly elliptic operator with formal adjoint \( A_m^\dagger \) and define

\[
B_{2m} = A_m^\dagger A_m = \sum_{\alpha:|\alpha| \leq m} \sum_{\beta:|\beta| \leq m} \tilde{c}_\alpha c_\beta (-1)^{|\alpha|} A^\alpha A^\beta
\]

on \( \mathcal{H} \). Since \( A^\alpha A^\beta = A^{\alpha + \beta} \) modulo lower-order terms, it follows that \( B_{2m} \) is a strongly elliptic operator. Moreover, if \( P_m \) denotes the principal part of the form \( C_m \) associated with \( A_m \) and \( P'_{2m} \) the principal part of the form \( C'_{2m} \) associated with \( B_{2m} \) then

\[
P'_{2m}(\xi) = |P_m(\xi)|^2 \geq (\text{Re } P_m(\xi))^2.
\]

Therefore one has the inequality \( p'_{2m} \geq p_m^2 \) for the ellipticity constants, with equality whenever the principal part of \( C_m \) is real. Now applying Theorem 1.2 to \( B_{2m} \) one deduces the following.

Corollary 3.1. Let \( A_m(c) \) be strongly elliptic. For each \( p \in (0, p_m(c)) \) there is a \( q \geq 0 \), independent of the representation, such that

\[
\|x\|_m \leq (1/p)\|A_m(c)x\| + q\|x\|
\]

for all \( x \in \mathcal{H}_\infty \). Consequently \( A_m(c) \) is closed on \( \mathcal{H}_m \) and the \( C^m \)-norm \( \| \cdot \|_m \) is equivalent to the graph \( x \mapsto \|A_m(c)x\| + \|x\| \).

Proof. Replacing \( A_m(c) \) by \( B_{2m} = A_m(c)^\dagger A_m(c) \) in (1.2) one finds for each \( p' \in (0, p'_{2m}) \) a \( q' \geq 0 \) such that

\[
p'\|x\|_m^2 \leq \|A_m(c)x\|^2 + q'\|x\|^2
\]

for all \( x \in \mathcal{H}_\infty \). But \( p'_{2m} \geq p_m^2 \). Thus if \( p \in (0, p_m) \) and \( p' = p^2 \),

\[
\|x\|_m^2 \leq (1/p^2)\|A_m(c)x\|^2 + q'\|x\|^2.
\]

Then (3.1) follows by elementary reasoning. But

\[
\|A_m(c)x\| \leq \left[ \sum_{\alpha:|\alpha| \leq m} |c_{\alpha}| \right] \|x\|_m
\]

and since \( \mathcal{H}_\infty \) is \( \| \cdot \|_m \)-dense in \( \mathcal{H}_m \) one immediately deduces the last statement of the corollary from (3.1) and (3.2).

Finally the foregoing reasoning extends to higher-order products. If \( A_{m_1}, \ldots, A_{m_n} \) are all strongly elliptic and \( m = m_1 + \cdots + m_n \) then

\[
B_{2m} = (A_{m_n}^\dagger \cdots A_{m_1}^\dagger)(A_{m_1} \cdots A_{m_n})
\]
is a strongly elliptic operator of order $2m$. Hence the same arguments show that the $C^m$-norm is equivalent to the graph norm

$$x \mapsto \| A_{m_1} \cdots A_{m_n} x \| + \| x \|.$$ 

Detailed proofs of these results will appear in [R2].

4. Semigroups bounds

The closure $\overline{A_m}$ of each strongly elliptic operator $A_m$ generates a strongly continuous semigroup $S$ holomorphic in a sector $\Delta_m(\phi) = \{ z \in \mathbb{C} ; \Re z > 0, |\Arg z| < \phi \}$ by Langlands' second theorem. Then if $\phi \in [0, \phi)$ it follows by general theory that there exist $M_\phi \geq 1$ and $\omega_\phi \geq 0$ such that

$$\| S_z \| \leq M_\phi e^{\omega_\phi |z|}$$

whenever $\Re z \geq 0$ and $|\Arg z| \leq \phi$. But the Gårding inequalities allow one to infer that $M_\phi = 1$, at least for small $\theta$.

Corollary 4.1. Let $C_m$ be a strongly elliptic form with ellipticity constant $p_m$, define

$$q_m = \sum_{\alpha : |\alpha| = m} |\Im c_\alpha|,$$

and $q_m = \tan^{-1} p_m / q_m$. Further let $S$ denote the holomorphic semigroup generated by $A_m(c)$.

If $\theta \in [0, \phi_m)$ then there is an $\omega_\theta \geq 0$ such that

$$\| S_z \| \leq e^{\omega_\theta |z|}$$

for all $z \in \mathbb{C}$ with $\Re z > 0$, and $|\Arg z| \leq \theta$.

Proof. First, by Langlands' estimates [L2], or by [R2], the semigroup $S$ is holomorphic in the sector $\Delta_m(\phi_m)$, and possibly in a larger sector. Thus if $z \in \mathbb{C}$ with $\Re z > 0$ and $|\Arg z| \leq \theta$ then $zA_m$ generates a continuous semigroup. But

$$\Re (x, zA_m x) = (\Re z)(\Re (x, A_m x) - |\Im z| |\Im (x, A_m x)|^2$$

$$\geq (\Re z) \Re (x, A_m x) - |\Im z| q_m \| x \|_{m/2}^2$$

$$- |\Im z| r_m \| x \|_{m/2} \cdot \| x \|_{m/2-1},$$

where

$$r_m = \sum_{\alpha : |\alpha| \leq m} |\Im c_\alpha|.$$

Now for each $\delta > 0$,

$$\| x \|_{m/2} \cdot \| x \|_{m/2-1} \leq \delta \| x \|_{m/2}^2 + (1/4\delta) \| x \|_{m/2-1}^2.$$

Moreover, for each $\sigma > 0$ there is a $k_\sigma > 0$ such that

$$\| x \|_{m/2-1} \leq \sigma \| x \|_{m/2}^2 + k_\sigma \| x \|^2.$$
Hence for each $\varepsilon > 0$ there is a $c_\varepsilon > 0$ such that

$$\text{Re}(x, z A_m x) = (\text{Re} z) \text{Re}(x, A_m x) - |\text{Im} z'(q_m + \varepsilon)||x||^2 m/2 - |\text{Im} z'| ||x||^2.$$  

Now we can use the second form of Gårding’s inequality (1.2) to deduce that for each $p' \in (0, p_m)$ there is a $q' > 0$ such that

$$\text{Re}(x, z A_m x) \geq (\text{Re} z) \text{Re}(x, A_m x) \left(1 - \frac{|\text{Im} z| q_m + \varepsilon}{p'}\right) - |\text{Im} z'| ||x||^2 \left(q'_m p' + c_\varepsilon\right).$$  

But by choosing $p'$ close to $p_m$ and $\varepsilon$ small, one can assure that $(1 - (q_m + \varepsilon)|\text{Im} z'|/p'\text{Re} z) > 0$. Then by another application of Gårding’s inequality there is a $q \geq 0$ such that $\text{Re}(x, A_m x) \geq -q||x||^2$. Therefore

$$\text{Re}(x, (z/|z|) A_m x) \geq -\omega_\theta ||x||^2$$

with $\omega_\theta = q\left(1 - ((q_m + \varepsilon)/p')\text{Tan} \theta\right) + q'_m p' + c_\varepsilon$.

Finally,

$$\frac{d}{d|z|} ||S_z x||^2 e^{-2\omega_\theta |z|} = -\text{Re}(S_z x, ((z/|z|) A_m + \omega_\theta I) S_z x) e^{-2\omega_\theta |z|} \leq 0.$$  

Therefore, by integration,

$$||S_z x|| \leq e^{\omega_\theta |z|} ||x||$$

for all $z \in \mathbb{C}$ with $\text{Re} z \geq 0$ and $|\text{Arg} z| \leq \theta$.

References


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