CAN THE WEYL ALGEBRA BE A FIXED RING?

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Abstract. If a finite soluble group acts as automorphisms of a domain, then the invariant subring is not isomorphic to the first Weyl algebra $\mathbb{C}[t,d/dt]$.

Let $R = \mathbb{C}[t,d/dt]$ be the first Weyl algebra. We prove the following result.

Theorem. Let $G$ be a finite solvable group. Let $S \supseteq R$ be a $\mathbb{C}$-algebra such that
(a) $S_R$ and $R_S$ are finitely generated,
(b) $S$ is a domain,
(c) $G$ acts as automorphisms of $S$, and $S^G = R$.
Then $S = R$.

We will prove a rather more general result, from which the theorem follows. The original proof was improved by comments of T. J. Hodges. I would like to thank him for his interest, and for allowing his improvements to be included here.

Let $B$ be an $R$-$R$-bimodule. We call $B$ an invertible bimodule, if there exists another bimodule, $C$ say, such that $B \otimes_R C$ is isomorphic to $R$ as a bimodule. The invertible bimodules form a group under the operation of tensor product $\otimes_R$; this group is called the Picard group, denoted $\text{Pic}(R)$. If $\sigma$, $\tau \in \text{Aut}(R)$ are $\mathbb{C}$-linear algebra automorphisms of $R$, then we write $\sigma R_\tau$ for the invertible bimodule which is $R$ as an abelian group, and for which the right $R$-module action is given by

$$ b \cdot x = b\tau(x) \quad \text{for } x \in R, b \in \sigma R_\tau $$

and the left $R$-module action is given by

$$ x \cdot b = \sigma(x)b \quad \text{for } x \in R, b \in \sigma R_\tau. $$

There is a map $\text{Aut}(R) \to \text{Pic}(R)$ given by $\sigma \mapsto \sigma R_1$. This is a group homomorphism. A key point in our analysis is the following result of J. T. Stafford.
Theorem [3, Corollary 4.5]. The map $\text{Aut}(R) \to \text{Pic}(R)$ is an isomorphism.

Hence if $B$ is an invertible $R$-bimodule, there exists $e \in B$ and $\sigma \in \text{Aut}(R)$ such that $x \cdot e = e \cdot \sigma(x)$ for all $x \in R$ (just take $e$ to be the image of 1 under the isomorphism $R_1 \to B$).

Proposition. Let $S \supseteq R$ be a $C$-algebra satisfying conditions (a) and (b) of the theorem. Then the only invertible $R$-bimodule contained in $S$ is $R$ itself.

Proof. Let $B \subset S$ be an invertible bimodule. Choose $e \in B$ and $\sigma \in \text{Aut}(R)$ such that $B = Re = eR$ and $x \cdot e = e \cdot \sigma(x)$ for all $x \in R$ (here $\cdot$ denotes multiplication in $S$). The multiplication in $S$ is an $R$-bimodule map, so $B^n \cong e^n R_1$. If $\sigma$ has infinite order (or equivalently, if $B$ has infinite order in $\text{Pic}(R)$), then all the bimodules $B^n$ are non-isomorphic, and their sum in $S$ would be direct. However, since $S_1$ is finitely generated, $S$ has finite length as an $R$-bimodule. Therefore, $\sigma^n = 1$ for some $n$. Hence for all $x \in R$, $xe^n = e^n \sigma^n(x) = e^n x$.

Therefore there is a surjective algebra homomorphism $R \otimes_C C[X] \to R[e^n]$, with $X \mapsto e^n$, where $X$ is an indeterminate commuting with $R$. By [1, 4.5.1], the ideals of $R \otimes_C C[X]$ are of the form $R \otimes_C I$ where $I$ is an ideal of $C[X]$. For $R[e^n] \cong R \otimes_C C[X]/R \otimes_C I \cong R \otimes_C C[X]/I$ to be a domain it is necessary that $I = \langle X - \alpha \rangle$ for some $\alpha \in C$. Thus $e^n = \alpha$. But $C[e] \subset S$ is a domain, so $n = 1$. Therefore $B = R$.

If $M$ is a left $R$-module, then the rank of $M$ is the dimension of $\text{Fract } R \otimes_R M$ as a left $\text{Fract } R$-module. It is clear that an invertible bimodule is of rank 1.

Proof of the theorem. First we prove it for $G$ abelian. In that case write $S = \bigoplus \chi S_{\chi}$, where the sum is over the irreducible characters of $G$, and $S_{\chi}$ is the $CG$-submodule of $S$ which is the sum of the $\chi$-isotypical components. Therefore $S_1 = R$, $S_\chi S_\xi = S_{\chi \xi}$, and each $S_\chi$ is an $R$-bimodule.

Suppose that $S_\chi = R$, and let $0 \neq a \in S_{\chi}$. Then $S_{\chi}a \subset R$, and isomorphic to $S_{\chi}$ as a left $R$-module since $S$ is a domain. In particular, $S_{\chi}$ is of rank 1 as a left $R$-module. Similarly, $S_\xi$ is of rank 1 as a right $R$-module. The multiplication map on $S$ gives an $R$-bimodule homomorphism $S_\chi \otimes_R S_{\xi} \to S_{\chi \xi}$. The image is non-zero subbimodule of $R$, hence equals $R$. Because all the ranks are 1, the map is injective. Therefore $S_{\chi}$ is an invertible bimodule. By the proposition, this forces $S_{\chi} = R$. Hence $S = R$ as required.

Now let $G$ be any finite solvable group, and set $H = [G, G]$. Then there is an action of $G/H$ as automorphisms of $S^H$, and $R = S^G = (S^H)^{G/H}$. But $G/H$ is abelian, and the first part of the argument applied to $S^H$ shows that $S^H = R$. Now by induction on $|G|$, the theorem follows.

Remarks. 1. It would be very nice to have the same result for an arbitrary finite group $G$, but a new idea is necessary. Not much is known about finitely generated $R$-bimodules which are not invertible, and that is probably a prerequisite.
2. I do not know of any domain $S \supset R$ such that (a) and (b) hold. It would be very interesting to know whether or not such an $S$ could exist. I expect not.

3. More generally I think it would be an interesting question to look at some other well understood non-commutative algebras, and ask if they can occur as the fixed ring of some reasonable extension ring. See [2] for an example concerning primitive factor rings of $U(sl(2))$.

References


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