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HOMOGENEOUS SPACETIMES OF ZERO CURVATURE

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ABSTRACT. In the following we show the only possible flat, connected, incomplete homogeneous spacetimes are H/Δ where $H = \{v \in \mathbf{R}^n | g(v, N) > 0\}$, N is a null vector, and Δ is a discrete subgroup of translations.

In [W1] Wolf classifies all complete flat homogeneous spacetimes. He does this by observing that the universal covering of such a spacetime must be Minkowski space, reducing the problem to finding all discrete subgroups of the Poincaré group which act properly discontinuously and whose normalizer acts transitively on Minkowski space. Classification of complete compact flat spacetimes in dimension 4 is given by Fried in [F]. In [W2] Wolf remarks that there are homogeneous, flat, incomplete spacetimes, and he gives an example. In this paper we will classify all incomplete, flat, homogeneous spacetimes. The first step in such a classification is to find all the possible universal covers of these spacetimes. Let N be a null vector in n dimensional Minkowski space. We show that the subspace H of n dimensional Minkowski space defined as $\{v: g(v, N) > 0\}$ is the unique (to within isometry), incomplete, homogeneous, simply connected, flat n dimensional spacetime (3.1). Next we find all discrete subgroups Δ of the isometry group of H which have transitive normalizers. Then every connected flat incomplete homogeneous spacetime is isometric to H/Δ. This result is given in Theorem 3.7.

Besides finishing the classification of all homogeneous flat spaces, this result relates to some other questions. Auslander ([Au, p. 809]) conjectured that every nilpotent, simply transitive group of affine transformations must contain a one parameter subgroup of central translations. This result was shown by Scheuneman [S, p. 226]. Auslander gives an example ([Au, p. 810]) of a three dimensional solvable group that acts affinely and simply transitively on $\mathbf{R}^3$ and contains no translations. A corollary of our result is that every subgroup of the Poincaré group having an open orbit in Minkowski space must contain a one

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parameter subgroup of translations (3.4). There are examples in every dimension of such subgroups of the Poincaré group whose translational part is one dimensional and non-central.

Also related to our result is the theorem that if G is a connected Lie group with a left invariant, flat, positive definite metric, then $[G, G]$ is abelian (see [M, p. 289]). We show every connected Lie group with a left invariant, flat, Lorentz metric is solvable (3.5). In the case of a unimodular group, this result follows from the known result that a connected Lie group with a left-invariant flat affine structure with parallel volume has a complete affine structure and is solvable.

Our result can be seen to fit in with the results of Yagi which give all the open homogeneous subsets of affine space. Let $V(r, n)$ be the Stiefel manifold whose elements consist of all sets of r independent vectors in $\mathbf{R}^n$. Yagi shows ([Y, p. 473]) that any connected, open, homogeneous subset, S, of affine space is a component of a product of Stiefel manifolds with an affine space. We show that if S is an orbit of a subgroup of the Poincaré group, then S is a component of $\mathbf{R}^{n-1} \times V(1, 1)$.

1. The developing map

Let M be a flat spacetime. In this section we list some of the properties of the developing map $\mathrm{dev}\colon \widetilde{M} \to \mathbf{R}^n$ where $\widetilde{M}$ is the universal cover of M, and $\mathbf{R}^n$ is Minkowski space. Details may be found in Fried, Goldman, Hirsch [FGH, p. 496] or [Y, p. 459].

Define $\mathrm{dev}\colon \widetilde{M} \to \mathbf{R}^n$ by $\mathrm{dev}(\tilde{m}) = (\mathrm{dev}_1(\tilde{m}), \dots, \mathrm{dev}_n(\tilde{m}))$ where $\{d(\mathrm{dev}_i)\}_{i=1}^n$ is a basis of covariant constant one forms on $\widetilde{M}$. We will need the following properties of the developing map.

Proposition 1.1. dev *is a local isometry.*

Let $I(\widetilde{M})$ denote the isometries of $\widetilde{M}$, and let $E^1(n)$ be the Poincaré group which is the isometry group of n dimensional Minkowski space.

Proposition 1.2. *There exists* $\alpha\colon I(\widetilde{M}) \to E^1(n)$ *defined by* $\mathrm{dev} \circ g = \alpha(g) \circ \mathrm{dev}$.

The following two results can be derived from [Y, 1.1, p. 459].

Theorem 1.3. *M_1 is isometric to M_2 if and only if there exists a mapping F from $\widetilde{M}_1$ to $\widetilde{M}_2$ such that F is an isometry with $F \circ \Delta_1 = \Delta_2 \circ F$ where Δ_i are the deck transformations of $\widetilde{M}_i \to M_i$. There exists $P \in E^1(n)$ such that $P \circ \mathrm{dev}_1 = \mathrm{dev}_2 \circ F$, and $P(S_1) = S_2$ where $S_i = \mathrm{dev}_i(M_i)$.*

When M is homogeneous we have the following stronger result concerning the developing map ([GH, p. 191]). This enables us to reduce the problem of finding all homogeneous simply connected spacetimes to finding the open homogeneous subspaces of Minkowski space.

Theorem 1.4. *If M is homogeneous, then* dev *is a covering projection.*

In fact it turns out we will need only the following:

Corollary 1.5. *If M is connected and homogeneous with $\Pi_1(\mathrm{dev}(\widetilde{M})) = 0$, then* dev *is a global isometry.*

2. Reduction to translation-free case

Let $S = \mathrm{dev}(\widetilde{M})$ in Minkowski space. G is the connected component of the identity of any transitive group of isometries acting on S. Let $\lambda(g)$ denote the linear part and $\tau(g)$ the translational part of $g \in G$. λ is a homomorphism and τ is a cocycle satisfying the following condition

$$\tau(g_1 g_2) = \lambda(g_1)\tau(g_2) + \tau(g_1).$$

Let $\mathrm{Ker}(\lambda) = \{\tau(g) \colon \lambda(g) = 1\}$. $\mathrm{Ker}(\lambda)$ is a subgroup of $\mathbf{R}^k$ since τ is a homomorphism when restricted to the group elements whose linear part is the identity. Let $\ker(\lambda)$ be the connected component of the identity of $\mathrm{Ker}(\lambda)$. $\ker(\lambda)$ is isomorphic to $\mathbf{R}^{n-k}$ for some k. In the following we will need information concerning the central elements of G.

Theorem 2.1. *Let G have an open orbit in $\mathbf{R}^n = V$. Let $z \in Z(G)$, the center of G. If $\lambda(z) \neq 1$, then*

$$\dim[\ker(1-\lambda(z)) \cap \ker(1-\lambda(z))^{\perp}] = 1.$$

Consequently, there is a null vector N which is an eigenvector of $\lambda(g)$ for all $g \in G$.

Proof. Assume $\ker(1-\lambda(z)) \cap \ker(1-\lambda(z))^{\perp} = 0$. Notice $\ker(1-\lambda(z))^{\perp} = \mathrm{Im}(1-\lambda(z))$. Therefore $V = \ker(1-\lambda(z)) \oplus \mathrm{Im}(1-\lambda(z))$.

Next we make a convenient change of cocycle. Let τ' be the original cocycle. Replace it by the following cohomologous cocycle

$$\tau(g) = \tau'(g) - \lambda(g)v + v$$

for some fixed $v \in V$ to be specified later. Apply this equation to z to find that

$$\tau(z) = \tau'(z) + (1-\lambda(z))v.$$

Let $\tau'(z) = w + x$ where $w \in \ker(1-\lambda(z))$ and $x \in \mathrm{Im}(1-\lambda(z))$. Let v be such that $x = -(1-\lambda(z))v$. This means $\tau(z) = w$ which is in $\ker(1-\lambda(z))$.

Now we use the fact that $z \in Z(G)$. We have

$$\begin{aligned}\tau(gz) &= \tau(g) + \lambda(g)\tau(z)\\ &= \tau(zg) = \tau(z) + \lambda(z)\tau(g).\end{aligned}$$

Thus

$$(1-\lambda(z))\tau(g) = (1-\lambda(g))\tau(z).$$

Now $\lambda(z)\lambda(g) = \lambda(g)\lambda(z)$ since λ is a group homomorphism. This means

$$(1-\lambda(z))(1-\lambda(g))\tau(z) = (1-\lambda(g))(1-\lambda(z))\tau(z) = 0.$$

Therefore, $(1-\lambda(z))^2\tau(g) = 0$, and

$$(1-\lambda(z))\tau(g) \in \ker(1-\lambda(z)) \cap \operatorname{Im}(1-\lambda(z)) = 0.$$

Thus, $\tau(g) \in \ker(1-\lambda(z))$ for all $g \in G$. Notice also that if $x \in \ker(1-\lambda(z))$, then

$$(1-\lambda(z))\lambda(g)x = \lambda(g)(1-\lambda(z))x = 0.$$

Hence, $\lambda(g)x \in \ker(1-\lambda(z))$ as well. Again let $x \in \ker(1-\lambda(z))$. Then

$$gx = \lambda(g)x + \tau(g) \in \ker(1-\lambda(z)).$$

Thus $G[\ker(1-\lambda(z))] = \ker(1-\lambda(z))$ which says that G acts by affine transformations on

$$V/\ker(1-\lambda(z)) \cong \ker(1-\lambda(z))^{\perp}.$$

G fixes 0 in this space, therefore acts by linear transformations. Also G preserves the induced metric on $\ker(1-\lambda(z))^{\perp}$ which is either positive definite or Lorentz. In either case G can not have an open orbit. Since the map $V \to V/\ker(1-\lambda(z))$ is open, V can not have an open orbit either. This gives the needed contradiction.

The following reduces the problem to finding all possible S with $\ker(\lambda)$ discrete.

Theorem 2.2. *For any $S \subseteq \mathbf{R}^n$ for which there is a subgroup G of the Poincaré group acting transitively on S there exists an integer $k \geq 0$ and an S' open in $\mathbf{R}^k$ such that $S \cong S' \oplus \mathbf{R}^{n-k}$. Furthermore, there is a group G' that acts affinely on $\mathbf{R}^k$ such that S' is an orbit of G'. G' satisfies one of the following properties*:

(A) *$\lambda(g)$ is a Lorentz transformation for all $g \in G'$.*

(B) *$J(\lambda(g)^* v^*, \lambda(g)^* w^*) = J(v^*, w^*)$ for all $g \in G'$ and $v^*, w^* \in \mathbf{R}^{k*}$ where J is an indefinite inner product on $\mathbf{R}^{k*}$ similar to* $\operatorname{diag}(0, 1, \ldots, 1)$.

Proof. Let W be a maximal linear subspace of $\mathbf{R}^n$ invariant under $\lambda(G)$ and satisfying $v + W \subseteq Gv$ for all $v \in \mathbf{R}^n$. There is a natural affine action of G on $\mathbf{R}^n/W$. Let $G_0 = \{g | gx = x \text{ for all } x \in \mathbf{R}^n/W\}$, $G' = G/G_0$. Let λ' and τ' denote the linear translational parts of the G' action. Notice that $\ker(\lambda') = 0$ since $\pi^{-1}(\ker\lambda')$ contains W, is $\lambda(G)$ invariant, and satisfies $v + \pi^{-1}(\ker\lambda') \subseteq Gv$. Now observe that S is isometric to $\pi(S) \times W$ because $s = \pi^{-1}\pi(S)$ by the second property of W. Let $S' = \pi(S)$ and $\mathbf{R}^k = \mathbf{R}^n/W$. We now only have to show that either (A) or (B) holds. There are two cases. The first is if $W \cap W^{\perp} = 0$. In this case $\mathbf{R}^n/W$ can be identified with $W^{\perp}$ which is invariant under $\lambda(G)$. Thus $\lambda(g)$ will be either a rotation or a Lorentz transformation depending on whether the spacetime metric on $\mathbf{R}^n$ when restricted

to $W^{\perp}$ is either positive definite or Lorentz. Since subgroups of the Euclidean group without translations have no open orbits, the positive definite case may be ignored, and we have case (A).

If $W \cap W^{\perp} = \operatorname{span}\{N\}$ where N is a null vector, there is a basis for W such that $W = \operatorname{span}\{N, v_2, \dots, v_{n-k-1}\}$ and $\mathbf{R}^n = \operatorname{span}\{N, v_2, \dots, v_n\}$. Define $V = \operatorname{span}\{v^*_{n-k}, \dots, v^*_n\}$. Then J in (B) is the dual metric when restricted to V.

Example 2.3. In $\mathbf{R}^3$ let G have Lie algebra $\operatorname{span}\{y\partial/\partial x - x\partial/\partial y + \partial/\partial t, \partial/\partial x, \partial/\partial y\}$. Then $\ker(\lambda) = \operatorname{span}\{\partial/\partial x, \partial/\partial y\}$, and $W = \mathbf{R}^3$.

Definition 2.4. G is of null type if there is a null vector which is an eigenvector of $\lambda(g)$ for every $g \in G$.

Theorem 2.5. *If G acts transitively on an open S, then G is of null type.*

The proof of this theorem will require the following lemmas.

Lemma 2.6. *If R is a compact connected solvable group, then R is a torus.*

Proof. See [Wa, p. 212, Theorem 6].

Lemma 2.7. *Let N be a connected nilpotent normal subgroup of a connected group G. Let K be a maximal torus in N. Then K is contained in the center of G.*

Proof. Let $g \in G$. Since $g^{-1}Ng = N$, we have $g^{-1}Kg$ is also a maximal compact subgroup of N. There is only one maximal compact subgroup of a nilpotent group [R, p. 6], therefore $g^{-1}Kg = K$. Define $\phi(g) \in \operatorname{Aut}(K)$ by $\phi(g)k = g^{-1}kg$. $\operatorname{Aut}(K) = \operatorname{Gl}(\mathbf{Z}, n)$ which is a discrete group. Since $\phi(G)$ is connected and contains the identity, we have $\phi(G) = \{I\}$. Therefore, every element of K is fixed under conjugation by elements of G. □

Lemma 2.8. *Let R be a connected solvable subgroup of* $\operatorname{SO}(n-1, 1)$. *Suppose there is a subspace W of V such that* $\dim(W) \leq 2$, $RW = W$, *and W contains a timelike vector. Then there is a torus T such that either $R = T$, or $R = T \oplus \mathbf{R}$.*

Proof. First assume $\dim W = 1$. Since R is connected and preserves length, it must fix any timelike vector in W. Thus $R \subseteq \operatorname{SO}(n-1)$, and hence R is compact. By Lemma 2.6, R is a torus. Next assume $\dim(W) = 2$. Let $\alpha\colon R \to \operatorname{SO}(1,1) \cong \mathbf{R}$ be defined by $\alpha(r) = r|_W\colon W \to W$. $\ker(\alpha)$ is a closed solvable subgroup of $\operatorname{Gl}(V)$ which is contained in a subgroup isomorphic to $\operatorname{SO}(n-2)$. This means $\ker(\alpha)$ is compact and by Lemma 2.6 is a torus. Thus, R has a torus of codimension at most one.

Next we will show that R is abelian. Let $N = K = \ker(\alpha)$ in Lemma 2.7. Then $\ker(\alpha)$ is in the center of R. Let $\mathfrak{r}$ be the Lie algebra of R and $\mathfrak{t}$ the Lie algebra of $T = \ker(\alpha)$. Since $\mathfrak{t}$ has codimension of at most 1 in $\mathfrak{r}$, there is a vector $r \in \mathfrak{r}$ such that for each $x, y \in \mathfrak{r}$ we have $x = t_1 + ar$, and $y = t_2 + br$ where $t_1, t_2 \in \mathfrak{t}$, and $a, b \in \mathbf{R}$. Since $\mathfrak{t}$ is in the center of $\mathfrak{r}$, we

have $[x, y] = 0$. Hence $\mathfrak{r}$ is abelian as desired. Since every connected abelian Lie group is a sum of its maximal torus with $\mathbf{R}^k$ for some k, [A, p. 15], the proof is complete. □

Lemma 2.9. *Let R be a connected solvable subgroup of* $\mathrm{SO}(n-1, 1)$ *and not of null type. Then there is a torus T such that R is either T or $T \oplus \mathbf{R}$.*

Proof. We need only produce a two dimensional R invariant W which contains a timelike vector, and then we can apply Lemma 2.8. Since R is not null, every R invariant subspace X has an R invariant complement $X^{\perp}$. Moreover, this complement is orthogonal to X. Therefore, the representation of R on V is completely reducible into a direct sum of irreducible orthogonal representations. Lie's theorem states that the dimension of an irreducible $\mathbf{R}$ representation of a solvable group is either 2 or 1 [R, p. 2]. Since V will be an orthogonal direct sum of irreducible subspaces, one of these subspaces must contain a timelike vector, and we are finished. □

Proof of Theorem 2.5. Assume G is not of null type. Use the reduction theorem to reduce the problem to the case when $\ker(\lambda)$ is discrete. Note that G of the reduced problem will not be of null type. Let R be the radical of $\lambda(G)$. First we will show that R must be either T or $T \oplus \mathbf{R}$ where T is some torus.

If R is not of null type, Lemma 2.9 gives this result. Assume R is of null type. Let N be a null vector such that for each $r \in R$ we have $r(N) = a(r)N$ for some $a(r) \in \mathbf{R}$. Since G is not of null type, there is a $g \in G$ such that $\lambda(g)N$ is independent of N. Let $W = \operatorname{span}\{N, N'\}$ where $N' = \lambda(g)N$. If W is $\lambda(G)$ invariant, then Lemma 2.8 gives this result. Therefore, assume $\lambda(G)W \neq W$. There is an element of G that takes one of the null vectors of W outside of W. Call this image $N'', N'' \notin W$. Note that all three of the null vectors N, N' and N'' are in the same G orbit. Also notice when N and xN are independent for some $x \in \lambda(G)$, the following is true.

$$r(xN) = x(x^{-1}rx)N = a(x^{-1}rx)xN.$$

If we let $b = a(x^{-1}rx)$ we find

$$\begin{aligned} bg(xN, N) &= g(rxN, N) \\ &= g(xN, r^{-1}N) \\ &= [a(r)]^{-1} g(xN, N). \end{aligned}$$

Thus, $b = [a(r)]^{-1}$ so $r(xN) = [a(r)]^{-1}xN$. This means $r(N') = [a(r)]^{-1}N'$ and $r(N'') = [a(r)]^{-1}N''$. Using the same argument with N' and N'' we find $a(r) = [a(r)]^{-1}$. Therefore, $a(r) = \pm 1$. Since R is connected, we have $a(r) = 1$. Thus the $\operatorname{span}\{N', N''\}$ is contained in the fixed point set of R which means it is R invariant. Also this space contains a timelike vector, and we may apply Lemma 2.8 to find our desired result.

We are now ready to finish the proof of the theorem. Use Lemma 2.7 to discover the torus T in R is in the center of $\lambda(G)$. Thus, $\lambda^{-1}(T)$ is central

in G since $\ker(\lambda)$ is discrete. Now use Theorem 2.1, and G is not of null type to conclude $T = 0$. Thus $R = 0$ or $\mathbf{R}$. If R is $\mathbf{R}$, then observe that $\mathbf{R}$ is in the center of $\lambda(G)$. This is true because $\lambda(G) = [\mathbf{R}]H$ where H is semisimple (Levi's Theorem). Consider the map $\phi\colon H \to \mathrm{Aut}(\mathbf{R}) \cong \mathbf{R}$. Since $\mathrm{Aut}(\mathbf{R})$ is abelian, we have ϕ induces a homomorphism from $H/[H,H] \to \mathrm{Aut}(\mathbf{R})$. $[H,H] = H$ because H is semisimple. Thus ϕ is trivial, and $\lambda(G) = \mathbf{R} \oplus H$ as desired. Now a similar argument to the one given above shows it is not possible to have $\mathbf{R}$ in the center of $\lambda(G)$. Thus, $\lambda(G) = H$ and is semisimple. By Whitehead's Lemma $H^1(\mathfrak{g}, V) = 0$, therefore the action of G on V is conjugate by a translation to a linear action. Since the full Lorentz group has no open orbits, this gives us our desired contradiction. □

3. Open orbits in the reduced case

Consider the two cases from the reduction theorem. We may pick a basis of $\mathbf{R}^n$ such that in (A) we find the Lie algebra for the linear part of G' using

$$J' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a subgroup of the following:

$$\mathfrak{g} = \left\{ \begin{bmatrix} \lambda & v & 0 \\ 0 & M & -v^T \\ 0 & 0 & -\lambda \end{bmatrix}, \lambda \in \mathbf{R}, v \in \mathbf{R}^{k-2}, M = -M^T \right\}.$$

For (B) by a simple calculation find

$$\mathfrak{g} = \left\{ \begin{bmatrix} U & v^T \\ 0 & \lambda \end{bmatrix}, U = -U^T, v \in \mathbf{R}^{k-1} \quad \text{and} \quad \lambda \in \mathbf{R} \right\}.$$

Therefore G' is a subgroup of

$$E_{k-1} \equiv \left\{ \begin{bmatrix} U & v^T \\ 0 & v \end{bmatrix}, U \in \mathrm{SO}(k-1), v \in \mathbf{R}^{k-1} \quad \text{and} \quad \lambda \in \mathbf{R} \right\}.$$

By the reduction theorem we need to consider two cases where G acts transitively on an open subset of $\mathbf{R}^k$. Define

$$H^{\pm} = \{v | v = (v_1, v_2, \ldots, v_k), \pm v_k > 0\}.$$

Theorem 3.1. *After an appropriate change of origin the open orbits of G are H^+ and H^-.*

Proof. Before proving the theorem, it is necessary to prove the following two lemmas. First we need to calculate the cocycles for case (A) and (B). We may first calculate cocycles on $\mathfrak{g}$. Since $\ker(\lambda)$ is discrete we have $\lambda_*\colon \mathfrak{g} \to \mathfrak{g}'$, where $\mathfrak{g}'$ is the Lie algebra of $\lambda(g)$, is an isomorphism. Define a cocycle θ on $\mathfrak{g}'$ by $\theta = \tilde{\tau}\lambda_*^{-1}$ where $\tilde{\tau} = D\tau$.

Lemma 3.2. *The cocycles of* $\mathfrak{g}$ *are*

$$\theta \begin{bmatrix} K & v \\ 0 & a \end{bmatrix} = \begin{bmatrix} w \\ 0 \end{bmatrix}$$

where $Kw = 0$.

Proof. E_{k-1} has a closed normal subgroup

$$H = \left\{ \begin{bmatrix} I & v \\ 0 & a \end{bmatrix}, v \in \mathbf{R}^{k-1}, a \in \mathbf{R} \right\}.$$

Let $N = G \cap H$. G/N is compact since $G/N \subseteq E_{k-1}/N \cong \mathrm{SO}(k)$ which is compact. Therefore, using the Hochschild-Serre spectral sequence and the fact that compact groups have trivial cohomology we have the following

$$0 \to H^1(G, V) \to H^1(N, V)^{G/N} \to 0$$

See Hochschild and Mostow [HM, Theorem 8.1] or Borel and Wallach [BW].

Let $\mathfrak{h} = \{\left[\begin{smallmatrix} 0 & v \\ 0 & a \end{smallmatrix}\right]\}$ be the Lie algebra of H. In the Lie algebra the cocycle and coboundary conditions are the following

$$\Xi[X, Y] = Y\Xi(X) - X\Xi(Y)$$

and $\Xi(X) = Xw_0$, where w_0 is a fixed vector.

Define

$$v_i = \begin{bmatrix} 0 & e_i \\ 0 & 0 \end{bmatrix}$$

for $i = 1, \dots, k-1$ and where $e_i = (0, \dots, 1, 0, \dots, 0)^T$;

$$v_k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By changing θ by a coboundary if necessary, we may assume $\theta(v_k) = (b, 0)^T$ where $b \in \mathbf{R}^{k-1}$. Using the cocycle condition we have

$$-\theta(v_i) = \theta[v_i, v_k] = v_k\theta(v_i) - v_i\theta(v_k).$$

Therefore

$$-\theta(v_i) = v_k\theta(v_i), \quad (1 + v_k)\theta(v_i) = 0.$$

Thus

$$\theta(v_i) = 0.$$

Hence a cocycle on $H^1(N, V)$ is $\theta(v_i) = (0, 0)^T$, $i = 1, \dots, k-1$, and $\theta(v_k) = (b, 0)^T$. We also need the cocycle to be G invariant. Define a G action on $Z^1(N, V)$.

$$g\theta(X) \equiv g^{-1}\theta(gXg^{-1});$$

then g acts on H^1 by $g[\theta] = [g\theta]$. A simple calculation shows for θ to be G invariant we must have $Kb = 0$ where $g = \left[\begin{smallmatrix} K & v \\ 0 & a \end{smallmatrix}\right]$ for all g in G. Next extend

the cocycle on $\mathfrak{h}$ by defining $\theta(g) = a(b,0)^T$ for $g = \begin{bmatrix} K & v \\ 0 & \lambda \end{bmatrix}$. Another calculation shows that the cocycle condition is satisfied, therefore none of the cocycles of $\mathfrak{g}$ have a nonzero k th component. □

Lemma 3.3. *The cocycles of* $\mathfrak{g}'$ *are*

$$\theta \begin{bmatrix} \lambda & v & 0 \\ 0 & K & -v \\ 0 & 0 & -\lambda \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

where $Kb = 0$ *and* $v \cdot b = 0$.

Proof. Similar to Lemma 3.2. □

Corollary 3.4. *In either case* (A) *or* (B), $\tau(g)$ *has* 0 *in the* k*th component for all* $g \in G$.

Proof. $\tilde{\tau}$ has 0 in the k th component since θ does. Since $\tau(1) = 0$ and G is connected, the group cocycle satisfies the same property. □

Proof of Theorem 3.1. In case (B) pick $(x,\varepsilon) \in \mathbf{R}^k$ such that (x,ε) is in an open orbit of G. Next let

$$G' = \left\{ g | g \in G \quad \text{and} \quad \lambda(g) = \begin{bmatrix} K & v \\ 0 & 1 \end{bmatrix} \right\}.$$

Observe that $G'(x,\varepsilon)^T = G(x,\varepsilon)^T \cap \{(z,\varepsilon)^T | z \in \mathbf{R}^{k-1}\}$ since G' acts by Euclidean transformations on $\{(z,\varepsilon) | z \in \mathbf{R}^{k-1}\}$ and $G'(x,\varepsilon)^T$ is open, we have $G'(x,\varepsilon) = \mathbf{R}^{k-1}$.

In case (A) define a mapping $\phi\colon \mathfrak{g} \to \mathbf{R}$ by

$$\phi \begin{bmatrix} \lambda & v & 0 \\ 0 & K & -v^T \\ 0 & 0 & -\lambda \end{bmatrix} = \lambda .$$

This is an algebra homomorphism where $\ker(\phi)$ is a subalgebra of the Lie algebra of the isotropy subgroup of a null vector. The orbits of such have codimension two in V. Since $\dim G \leq 1 + \dim(\ker(\phi))$, G can have orbits at most of codimension 1. Thus in this case we have no open orbits. □

Corollary 3.5. *Any subgroup of the Poincaré group which acts transitively on Minkowski space contains a translation.*

Proof. Suppose G has no translations. Then we have case (A) from Theorem 2.2. Therefore, G has no open orbits. The result follows from this contradiction.

Corollary 3.6. *A Lie group with left invariant flat spacetime metric must be solvable.*

Proof. Let G be a Lie group with the above metric. Consider the universal cover of G denoted by $\widetilde{G}$. By (3.3) $\widetilde{G}$ is diffeomorphic to $\mathbf{R}^n$ or H. Both of these are homeomorphic to $\mathbf{R}^n$. Therefore, $\widetilde{G}$ and G are both solvable.

Theorem 3.7. *Let M be a connected homogeneous flat spacetime. Let $\mathbf{R}^n$ denote Minkowski space. Let N be any null vector in $\mathbf{R}^n$. One of the following is true.*

(i) *M is complete and by* [Wl, p. 466] *or* [W3, p. 135] *M is isometric to $\mathbf{R}^n/\Delta$ where Δ is a discrete subgroup of the group of pure translations (which we identify with $\mathbf{R}^n$).*

(ii) *M is incomplete. There exists a discrete subgroup Δ of the translation group such that*
   (a) $\Delta \subseteq N^\perp$,
   (b) $N \notin \operatorname{span} \Delta$,
   (c) $M \cong H$ *where* $H = \{v | g(v, N) > 0\}$.

Moreover, $H/\Delta \cong H/\Delta'$ if and only if Δ is conjugate to Δ' by a Lorentz transformation that has N as an eigenvector.

Proof. Let $\delta = T_v A \in \Delta$. By Wolf [W, p. 132] we have $\operatorname{Im}(A - I)$ consists entirely of null vectors and hence $\dim \operatorname{Im}(A - I) \leq 1$. First we show $\operatorname{Ker}(A - I) \subseteq \operatorname{Im}(A - I)^\perp$. Suppose $y \in \operatorname{Ker}(A - I)$. Then

$$\begin{aligned} g((A - I)x, y) &= g(Ax, y) - g(x, y) \\ &= g(x, A^{-1}y) - g(x, y) \\ &= 0. \end{aligned}$$

Since $\dim \operatorname{Ker}(A-I) \geq n-1$ and $\operatorname{Ker}(A-I) \subseteq \operatorname{Im}(A-I)^\perp$ we have $\operatorname{Ker}(A-I) = \operatorname{Im}(A - I)^\perp$. $\operatorname{Im}(A - I)$ is null, therefore $\operatorname{Im}(A - I) \subseteq \operatorname{Im}(A - I)^\perp$. Thus we have $\operatorname{Ker}(A - I) \supseteq \operatorname{Im}(A - I)$.

Let N' be a generator of $\operatorname{Im}(A - I)$. Then $(A - I)x = f(x)N$ where f is linear. $A^{-1}x = x - f(x)N'$ therefore $g(x, x) + f(x)g(N', x) = g(x, Ax) = g(A^{-1}x, x) = g(x, x) - f(x)g(N', x)$ for all x on an open set. Since f is linear we have $f = 0$ and $A - I = 0$.

We now observe for M to be homogeneous then $N(\Delta)$ must be transitive on H and thus must have an open orbit on $\mathbf{R}^n$. Therefore $N(\Delta)_0 = Z(\Delta)$ the centralizer of Δ must also have an open orbit. Let $z \in Z(\Delta)$. A simple calculation shows z is linear and $z(v) = v$.

Suppose $N \in \operatorname{span}(\Delta)$. Let $w \in H_N$ with $g(w, N) \neq 0$; then $z(w) = \lambda w + x$, $x \in N^\perp$. Thus

$$g(N, w) - g(z^{-1}(N), w) = g(N, z(w)) = \lambda g(N, w) + g(w, x) = \lambda g(N, w).$$

Therefore, $\lambda = 1$ and $(A - I)w \in N^\perp$ and we have no open orbits.

If $N \notin \operatorname{span}(\Delta)$, *i.e.* $\operatorname{span}(\Delta)$ is spacelike, then we need to find $z \in Z(\Delta)$ such that $z(v) = v$ for all $v \in \Delta$ and $z(w + N^\perp) \neq w + N^\perp$ for some w.

Let $\operatorname{span}(\Delta) = W \subseteq N^\perp$. Then $W^\perp$ is a spacetime with $\dim W^\perp \geq 2$. There exists $\widetilde{N}$ such that $\operatorname{span}\{N, \widetilde{N}\} = X \subseteq W^\perp$ and $\dim X = 2$. $X \oplus X^\perp = V$ with $X^\perp \supseteq W$. Define a Lorentz transformation $A(N) = \frac{1}{2}N$, $A(\widetilde{N}) = 2\widetilde{N}$, $A(y) = y$, $y \in X^\perp$. Then $A(\widetilde{N} + N^\perp) = 2\widetilde{N} + N^\perp$ and we have the result.

By Theorem 1.3 we have $H/\Delta \cong H/\Delta'$ if and only if Δ is conjugate to Δ' by a Poincaré transformation $h \in I(H)$. Therefore h is a Lorentz transformation which has N as an eigenvector.

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