ALGEBRAIC STRUCTURE
IN COMPLEX FUNCTION SPACES

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Abstract. Let $M$ be a complex function space containing constants, and let $Z$ be the complex state space of $M$. If $M$ is linearly isometric to a uniform algebra and if $Z$ is affinely homeomorphic to the complex state space of a uniform algebra then we prove that $M$ is a uniform algebra. Neither of the two conditions taken separately imply this conclusion.

If $X$ is a compact Hausdorff space then $C_c(X), C_R(X)$ will denote the Banach spaces of all continuous complex-valued, respectively real-valued, functions on $X$ with the supremum norm. A closed linear subspace $M$ of $C_c(X)$ which contains constants and separates the points of $X$ will be called a complex function space. The subset $S = \{\phi \in M^* : \|\phi\| = 1, \phi(1)\}$ of $M^*$ is called the state space of $M$ and the subset $Z = \text{co}(S \cup -iS)$ of $M^*$ is called the complex state space of $M$; the sets $S$ and $Z$ are compact convex sets when endowed with the relative $w^*$-topology. If $K$ is any compact convex subset of a locally convex Hausdorff space then $A(K), A_c(K)$ will denote the Banach spaces of all continuous real-valued, respectively complex-valued, affine functions on $K$ with the supremum norm.

We shall be concerned with the linear and norm structure of $M$ and the affine and topological structure of $Z$ and will seek conditions which imply that $M$ is a uniform algebra on $X$. To this end we say that the complex state spaces $Z_1, Z_2$ of two complex function spaces $M_1, M_2$ are equivalent if they are affinely homeomorphic, and that $Z_1, Z_2$ are real-equivalent if there is an affine homeomorphism $\eta : Z_1 \rightarrow Z_2$ which maps $S_1$ onto $S_2$ (and hence maps $-iS_1$ onto $-iS_2$).

We begin by developing [4, Examples 3 and 1] to show that the property that $M$ has the linear and norm structure of a uniform algebra is independent from the property that $Z$ is equivalent (or real-equivalent) to the complex state space of a uniform algebra.
Example 1. Let $M_1 = P(\Gamma)$ be the disc algebra on the unit circle $\Gamma$, and let $M = \{zf(z) : f \in M_1\}$. Then [4, Example 3] shows that $M$ and $M_1$ are isometrically isomorphic while $Z$ and $Z_1$ are not equivalent. We will show that $Z$ is not equivalent to $Z_2$ for any uniform algebra $M_2$.

Suppose that $Z$ is equivalent to $Z_2$. Then the connectedness of $\Gamma$ implies that either $Z$ is real-equivalent to $Z_2$ or $Z$ is real-equivalent to the complex state space of $M_2$ (cf. [5]). We may hence assume that $Z$ and $Z_2$ are real-equivalent, and that $M_2$ is a uniform algebra on $\Gamma$ (cf. [4]). We have $M_2 = \{u + iv \circ \psi : u + iv \in M\}$, where $\psi : \Gamma \to \Gamma$ is a homeomorphism.

Now $z$ and $\bar{z}$ belong to $M$ and so the functions $f(z) = x + \text{im} y(z)$ and $\bar{f}(z) = x - \text{im} y(z)$ belong to $M_2$, where we write $z = x + iy$. Hence the function $g(z) = x$ belongs to $M_2$ and similarly, using the facts that $iz$ and $i\bar{z}$ belong to $M$, we see that the function $h(z) = y$ belongs to $M_2$. Consequently the uniform algebra $M_2$ equals $C_c(\Gamma)$. This implies that $M = C_c(\Gamma)$, giving the required contradiction.

Example 2. Let $M_1 = P(\Gamma)$ and $M = \{f : f(z) = u(z) + iv(-z) : u + iv \in M_1\}$. Then [4, Example 1] shows that $Z$ and $Z_1$ are real-equivalent while $M$ and $M_1$ are not isometrically isomorphic. We will show that $M$ is not isometrically isomorphic to any uniform algebra $M_2$.

If $M$ is isometrically isomorphic to $M_2$, a uniform algebra on $X$, then we will have $M_2 = \{\lambda(f \circ \tau) : f \in M\}$, where $\lambda \in M_2$ with $|\lambda| = 1$ and $\tau : X \to \Gamma$ a homeomorphism. Writing $M_3 = \{g \circ \tau^{-1} : g \in M_2\}$ we note that $M_3$ is a uniform algebra on $\Gamma$ equal to $\{(\lambda \circ \tau^{-1})f : f \in M\} = \{lf : f \in M\}$, where $l = \lambda \circ \tau^{-1} \in M_3$ with $|l| = 1$. Since $z^{2n}$ and $\bar{z}^{2n+1}$ belong to $M$, $n \geq 0$, the functions $l(z)z^{2n}$ and $l(z)\bar{z}^{2n+1}$ belong to $M_3$. Since $M_3$ is an algebra and since $l(z)^2z^{2n} = l(z)(l(z)z^{2n})$, $l(z)^2\bar{z}^{2n+1} = l(z)(l(z)\bar{z}^{2n+1})$, $l(z)^2\bar{z}^{2n+1} = (l(z)z)(l(z)\bar{z}^{2n+2})$, $l(z)^2z^{2n} = (l(z)\bar{z})(l(z)\bar{z}^{2n+1})$ we see that the functions $l(z)^2z^k$, $k$ any integer, belong to $M_3$. Since the polynomials in $z$ and $\bar{z}$ form a dense linear subspace of $C_c(\Gamma)$ it follows that $M_3 = C_c(\Gamma)$. This implies that $M = C_c(\Gamma)$, giving the required contradiction.

We will now show that if $M$ has the linear and norm structure of a uniform algebra, and if $Z$ is equivalent to the complex state space of a uniform algebra, then $M$ is necessarily a uniform algebra. We note firstly however that we cannot replace ‘complex state space’ by ‘state space’ in this result. Indeed, in Example 1 above $M$ and $M_1$ are isometrically isomorphic and, since $M$ contains the Dirichlet algebra $M_1$, the state spaces of $M$ and $M_2$ are equivalent to the state space of $C_R(\Gamma)$.

We need to recall some concepts, full details of which may be found in Asimow and Ellis [1]. The centre of $A(K)$ consists of those functions $f \in A(K)$ such that for each $G \in A(K)$ there is some $h \in A(K)$ satisfying $h(x) = f(x)g(x)$ for all $x \in \partial K$, where $\partial K$ denotes the set of extreme points of $K$. The sets of constancy in $\partial K$ for the central functions in $A(K)$ form...
the sets of extreme points of a family of faces \( \{ F_\alpha \} \) of \( K \), called the \textit{Šilov decomposition} for \( A(K) \). The maximal subsets \( E \) of \( \partial K \) such that the centre of \( A(\overline{\partial K}) \) is trivial form the sets of extreme points of a family of faces \( \{ F_\beta \} \) of \( K \) called the \textit{Bishop decomposition} for \( A(K) \). In the case when \( K \) is the complex state space of a uniform algebra these decompositions are closely related to the corresponding classical decompositions.

If \( Z \) is the complex state space of a function space \( M \) then \( \theta : M \to A(Z) \) will denote the real-linear homeomorphism defined by \( \theta f(z) = \text{re } z(f) \), noting that \( \theta(u + iv)(\lambda x - i(1 - \lambda)y) = \lambda u(x) + (1 - \lambda)v(y) \) when \( x, y \in X \) and \( 0 \leq \lambda \leq 1 \). For this purpose we consider \( X \) to be canonically embedded in \( S \). \( \theta_1, \theta_2 \) will denote the corresponding maps for \( M_1 \) and \( M_2 \).

**Theorem 1.** Let \( M \) be a complex function space on \( X \) with complex state space \( Z \), and let \( M_j \) be uniform algebras with complex state spaces \( Z_j, j = 1,2 \). If \( M \) is isometrically isomorphic to \( M_1 \) and if \( Z \) is equivalent to \( Z_2 \) then \( M \) is a uniform algebra on \( X \).

**Proof.** We first prove the result in the special case when \( Z \) is real-equivalent to \( Z_2 \).

As in the discussion of the Examples above we may assume that \( M_1, M_2 \) are uniform algebras on \( X \), and that

\[
M = \{ l f : f \in M_1 \} = \{ u + iv \circ \psi : u + iv \in M_2 \},
\]

where \( l \in M \) with \( |l| = 1 \) and \( \psi : X \to X \) is a homeomorphism with \( \psi^2 \) equal to the identity map on the essential set for \( M_2 \). In order to prove that \( M \) is an algebra it will be sufficient to show that \( l \in M_1 \), that is \( l^2 \in M \). Write \( l = g + ih \) so that \( g + ih = u + iv \circ \psi \) for some \( u + iv \in M_2 \). Since \( M \) contains constants we must have \( l \in M_1 \), so that \( l = l^2 \in M \). Hence \( g - ih = u_1 + iv_1 \circ \psi \) for some \( u_1 + iv_1 \in M_2 \). Consequently we obtain \( g = u = u_1, h = v \circ \psi = -v_1 \circ \psi \), so that \( v = -v_1 \) and \( u - iv, u \) and \( v \) belong to \( M_2 \). But then \( l^2 = g^2 - h^2 + 2igh = 2u_2^2 - 1 + 2i((u \circ \psi^{-1})v) \circ \psi \) belongs to \( M \) because \( u \circ \psi^{-1} \) belongs to \( M_2 \) (cf. [4]). Hence \( M \) is an algebra.

We now turn to the general case where \( Z \) and \( Z_2 \) are equivalent and \( M = \{ l f : f \in M_1 \} \), with \( l \in M \). Firstly we identify the centres of \( A(Z) \) and \( A(Z_1) \). The centre of \( A(Z_1) \) consists of the functions \( \theta_1(u + iv) \) such that \( u, v \) belong to \( M_1 \) and \( u - v \) belongs to the essential ideal for \( M_1 \) (cf. [2, Theorem 1]).

Suppose that \( \theta(u + iv) \) belongs to the centre of \( A(Z) \). Then for each \( a + ib \in M \) we have \( ua + ivb \) belongs to \( M \), and since 1, i belong to \( M \) we may deduce that \( u \) and \( v \) belong to \( M \). If \( a + ib \in M \) then we have \( b - ia \in M \) and hence \( ub - iva \) and \( va + iub \) belong to \( M \). Consequently \( (a + ib) \) belongs to \( M \) and \( (u + ivb - va - iub) = (u - v)(a - ib) \) belongs to \( M \). Conversely, reversing this argument, we see that \( \theta(u + iv) \) belongs to the centre of \( A(Z) \) whenever \( (u + v)M \) and \( (u - v)\overline{M} \) are contained in \( M \).
Therefore \( \theta(u+iv) \) belongs to the centre of \( A(Z) \) if and only if \( f \in M_1 \) implies that \((u+v)f\) and \((u-v)\overline{f} \) belong to \( M \), that is \((u+v)f\) and \((u-v)\overline{f} \) belong to \( M_1 \). Taking \( f = 1 \) and also \( f = \overline{f} \in M_1 \), we see that \( u+v, u-v, u \) and \( v \) belong to \( M_1 \) whenever \( \theta(u+iv) \) belongs to the centre of \( A(Z) \). In this case taking \( f = \overline{g} \), where \( g \in M_1 \), we see that \((u-v)g \in M_1 \); since \((u-v)g \) belongs to \( M_1 \) it follows that \((u-v)\overline{g} \) and \((u-v)\overline{g} \) belong to \( M_1 \). The proof of [2, Theorem 1] now shows that \( u-v \) belongs to the essential ideal \( I_1 \) of \( M_1 \), that is \( \theta_1(u+iv) \) belongs to the centre of \( A(Z_1) \). Conversely, if \( u, v \in M_1 \), and \( u-v \in I_1 \) then, for all \( f \in M_1 \), we have \((u+v)f, (u-v)\overline{f} \in M_1 \), because \( \overline{f} \in C_\alpha(X) \). Hence \( \theta(u+iv) \) belongs to the centre of \( A(Z) \), and we have shown that the centres of \( A(Z) \) and \( A(Z_1) \) may be identified.

We may assume without loss of generality that \( X \) is the Šilov boundary for both \( M_1 \) and \( M \). Therefore we have shown that the Šilov decompositions of \( Y = X \cup -iX \) corresponding to \( A(Z) \) and \( A(Z_1) \) coincide. The Šilov decompositions of \( Y \) for \( A(Z_1) \), except for the singleton sets, consists of sets of the form \( E_\alpha \cup -iE_\alpha \), where \( E_\alpha \) belongs to the Šilov decomposition of \( X \) for \( M_1 \) (cf. [3]). Now \( M_1|E_\alpha \) and \( M|E_\alpha \) are isometrically isomorphic, and we may apply the preceding reasoning to these spaces to conclude that the Bishop decompositions of \( Y \) corresponding to \( A(Z) \) and \( A(Z_1) \) coincide.

The Bishop decomposition for \( A(Z) \), except for singletons, consists of faces of the form \( G_\beta = co(F_\beta \cup -iF_\beta) \), where \( F_\beta \cap X = E_\beta \) belongs to the Bishop decomposition for \( M_1 \). Moreover, if \( g \in C_\alpha(X) \) is such that \( g|E_\beta \) belongs to \( M|E_\beta \) for all \( \beta \), then \( \overline{g} \in C_\alpha(X) \) and \( \overline{g}|E_\beta \subseteq M_1|E_\beta \) for all \( \beta \) which implies that \( \overline{g} \in M_1 \), and hence \( g \) belongs to \( M \). We can hence conclude that \( M \) is an algebra if we can show that \( \overline{f}|E_\beta \) belongs to \( M|E_\beta \) for all \( \beta \).

Since \( Z \) is equivalent to \( Z_2 \), the faces of the Bishop decompositions for \( Z \) and \( Z_2 \) are equivalent. Therefore if we restrict attention to \( M|E_\beta \) and \( M_1|E_\beta \) we see that the complex state space \( G_\beta \) of \( M|E_\beta \) is equivalent to the complex state space of an antisymmetric uniform algebra \( M_3 \) (a restriction algebra of \( M_2 \)). However in this case either \( G_\beta \) is real-equivalent to \( Z_3 \) or is real-equivalent to the complex state space of \( M_3 \). In either case the first part of the proof shows that \( M|E_\beta \) is an algebra. Consequently \( \overline{f}|E_\beta \) belongs to \( M|E_\beta \) and the proof of the theorem is complete.

We remark that the condition in Theorem 1 that \( Z \) is equivalent to \( Z_2 \) is much weaker than the condition that \( Z \) is real-equivalent to \( Z_2 \). In fact if \( M \) is self-adjoint and if \( Z \) is real-equivalent to \( Z_2 \) then, since \( S \) is a split face of \( Z \), \( S_2 \) must be a split face of \( Z_2 \) which implies that \( M_2 \) is a \( C_\alpha(X) \)-space. This conclusion need not hold when \( Z \) and \( Z_2 \) are just equivalent, as the following example shows.

**Example 3.** Let \( Z_\Gamma, Z_{\Gamma'} \) denote respectively the complex state spaces of \( P(\Gamma) \), \( P(\Gamma') \), where \( P(\Gamma') \) is the uniform algebra generated by the polynomials on
\( \Gamma' = \{ z \in \mathbb{C} : |z - 3| = 1 \} \). Let \( M = A_C(Z_\Gamma) \) and \( M_2 = P(\Gamma \cup \Gamma') \). Then \( Z_2 \) is the convex hull of the disjoint closed split faces \( Z_\Gamma \) and \( -iZ_\Gamma \), while \( Z \) is the convex hull of the disjoint closed split faces \( Z_\Gamma \) and \( -iZ_\Gamma \). Since \(-iZ_\Gamma \) and \( Z_\Gamma \) are equivalent so are \( Z \) and \( Z_2 \). In this example \( M \) is self-adjoint while the uniform algebra \( M_2 \) is not a \( C_C(X) \)-space.

We note that it is easy to verify that no non-trivial uniform algebra can be isometrically isomorphic to a self-adjoint complex function space.

In the context of Theorem 1, Nagasawa's theorem [7] show that \( M_1 \) is unique in the sense that any two isometrically isomorphic uniform algebras are algebraically isomorphic. On the other hand \( M_2 \) need not be unique even if \( M \) is a \( C_C(X) \)-space (cf. [4, Example 2]). Our final result gives conditions under which \( M_2 \) is uniquely determined, up to complex conjugation. A related result appeared in Ellis and So [5, Corollary 6]).

**Theorem 2.** Let \( M_1, M_2 \) be uniform algebras with essential sets \( X, Y \) respectively. If \( Z_1 \) and \( Z_2 \) are equivalent then \( M_2|Y \) is isometrically isomorphic to \((M_1|E) \otimes \overline{M_1}|X\setminus E)\), for some open and closed subset \( E \) of \( X \).

**Proof.** Let \( \varphi : Z_1 \to Z_2 \) be an equivalence. The essential face for \( Z_1 \) has the form \( \text{co}(F \cup -iF) \), where \( F \) is the closed convex hull of \( X \) in \( S_1 \) (cf. [3, Proposition 17]). Since \( \varphi \) maps the essential face of \( Z_1 \) onto the essential face of \( Z_2 \), and since \( \text{co}(F \cup -iF) \) is the complex state space of \( M_1|X \), we can assume without loss of generality that \( M_1 \) and \( M_2 \) are essential uniform algebras, and that \( X, Y \) are the Šilov boundaries of \( M_1, M_2 \) respectively.

If we write \( E = \{ x \in X : \varphi(x) \in S_2 \} \) then \( X = E \cup (X\setminus E) \) is a peak-set decomposition of \( X \) for \( M_1 \) (cf. [5, Corollary 2]). Since \( M_1 \) is essential so are the algebras \( M_1|E \) and \( M_1|(X\setminus E) \) and hence \( E \) (respectively \( X\setminus E \)) is the closure of the union of non-singleton maximal antisymmetric sets for \( M_1|E \) (respectively \( M_1|(X\setminus E) \)) (cf. [6, page 65]).

The faces of the Bishop decomposition for \( Z_1 \) are the singletons \( x, -ix \), where \( x \) is a singleton member of the Bishop decomposition for \( M_1 \), together with faces of the form \( \text{co}(F_n \cup -iF_n) \), where \( F_n \) is the closed convex hull in \( S_1 \) of a non-singleton member of the Bishop decomposition for \( M_1 \). Now each \( \text{co}(F_n \cup -iF_n) \) is mapped by \( \varphi \) onto a corresponding member \( \text{co}(G_n \cup -iG_n) \) of the Bishop decomposition for \( Z_2 \). Consequently \( \text{co}(E \cup -iE) \) is mapped onto a face of the form \( \text{co}(H \cup -iH) \), and similarly for \( \overline{\text{co}}((X\setminus E) \cup -i(X\setminus E)) \). Since the Bishop decompositions determine \( M_1 \) and \( M_2 \), and since we have \( M_1|(F_n \cap X) = \{ f \circ \varphi : f \in M_2|(G_n \cap Y) \} \) whenever \( F_n \cap E \) is non-empty we see that \( M_1|E = \{ f \circ \varphi : f \in M_2|\varphi(E) \} \). Therefore \( M_1|E \) is isometrically isomorphic to \( M_2|\varphi(E) \). Similarly we may prove that \( M_1|(X\setminus E) \) is isometrically isomorphic to \( M_2|(Y\setminus \varphi(E)) \).
REFERENCES


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