THE TORUS LEMMA ON CALIBRATIONS, EXTENDED

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Abstract. The whole face $G(\phi)$ of $m$-planes calibrated by a torus $m$-form $\phi$ is determined by the torus face $G_T(\phi)$. Indeed, $G(\phi)$ results from applying a new closure operation to $G_T(\phi)$.

1. Introduction

Over the past ten years the theory of calibrations has illuminated the occurrence and structure of singularities in $m$-dimensional area-minimizing surfaces. This note gives an extension of a much-used lemma on calibrations, the Torus Lemma (cf. §3). Our observations bear on recent work of D. Nance [N, e.g. Corollary 3.8] and M. Messaoudene [Me].

For surveys on calibrations see [H1], [M1], [M2]. For basic concepts and definitions see [M3, §1, §2], [M4, Chapter 4], the original paper [HL], or the new text [H2].

2. Definitions

In addition to the standard dual Euclidean norms on the exterior algebra $\Lambda^m \mathbb{R}^n$ and its dual $\Lambda^m \mathbb{R}^{n*}$, there is another important dual pair of norms, called mass and comass. The comass $\|\phi\|^*$ of a form $\phi \in \Lambda^m \mathbb{R}^n$ is the maximum value of $\phi$ on the Grassmannian $G(m, \mathbb{R}^n)$ of oriented unit $m$-planes through 0 in $\mathbb{R}^n$:

$$\|\phi\|^* = \max\{\langle \xi, \phi \rangle : \xi \in G(m, \mathbb{R}^n)\}.$$  

A form $\phi$ normalized to have comass 1 is called a calibration. The face $G(\phi)$ of a calibration $\phi$ consists of its maximum points in the Grassmannian:

$$G(\phi) = \{\xi \in G(m, \mathbb{R}^n) : \langle \xi, \phi \rangle = 1\}.$$  

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The mass is the dual norm on \( m \)-vectors \( \xi \) in \( \Lambda^m \mathbb{R}^n \):

\[
||\xi|| = \max\{\langle \xi, \varphi \rangle : ||\varphi||^* = 1\}
= \min\left\{\sum a_j : \xi = \sum a_j \xi_j, \xi_j \in G(m, \mathbb{R}^n), a_j > 0\right\}.
\]

A calibration \( \varphi \) for which the maximum of \( \langle \xi, \varphi \rangle \) is attained is said to calibrate \( \xi \). An expression \( \xi = \sum a_j \xi_j \) for which the minimum is attained is called a mass decomposition for \( \xi \). A calibration \( \varphi \) calibrates \( \xi = \sum a_j \xi_j \) if and only if all the \( \xi_j \) lie in the face \( G(\varphi) \).

In the case that \( n = 2m \) and \( \mathbb{R}^n = (\mathbb{R}^2)^m \) consider the \( m \)-dimensional torus

\[
T = (S^1)^m = (G(1, \mathbb{R}^2))^m \subset G(m, \mathbb{R}^{2m}) \subset \wedge^m \mathbb{R}^{2m}.
\]

The elements of \( T \) are called torus planes. Let \( T_S \) denote the span of \( T \) in \( \Lambda^m \mathbb{R}^{2m} \). The torus span \( T_S \) can also be described as the tensor product \( T_S = \bigotimes_{i=1}^m \wedge^1 \mathbb{R}^2 \). Elements of \( T_S \) are called torus \( m \)-vectors. Similarly the elements of the dualspace \( T_S^* \) are called torus forms. The intersection of the face \( G(\varphi) \) of any calibration \( \varphi \) with the torus \( T \) is called the torus face \( G_T(\varphi) \).

3. The Torus Lemma

The Torus Lemma ([M5, Lemma 4], cf. [DHM, §4]) says that a torus calibration attains its maximum value of 1 on the torus. Equivalently,

\[
\left\{\varphi \in T_S^* : \max_{\xi \in T} \langle \xi, \varphi \rangle = 1\right\} = \left\{\varphi \in T_S^* : ||\varphi||^* = 1\right\}.
\]

We observe a few useful consequences.

(1) The unit mass ball intersects the torus span in the convex hull of \( T \).

(2) A torus \( m \)-vector is calibrated by a torus calibration [Me, 4.4.2].

(3) A torus \( m \)-vector has a mass decomposition in terms of torus \( m \)-planes. (This generalizes [Me, 5.2.8, 5.4.3].)

(4) Let \( \varphi \) be a torus calibration. An \( m \)-plane \( \xi \) belongs to \( G(\varphi) \) if and only if its projection \( P\xi \) onto the torus span \( T_S \) is a convex combination of \( G_T(\varphi) \).

Consequence (4) includes the new observation that the face \( G(\varphi) \) of a torus form \( \varphi \) is determined by the torus face \( G_T(\varphi) \). This fact is applied by D. Nance [N, e.g. Corollary 3.8]. Theorem 6 will exhibit this fact in another way.

The consequences follow immediately from (*) . For example, we will verify (4). Since \( \varphi \) is a torus calibration, \( \varphi(\xi) = \varphi(P\xi) \). Clearly if \( P\xi \) is a convex combination of \( G_T(\varphi) \), then \( \varphi(\xi) = \varphi(P\xi) = 1 \), so that \( \xi \in G(\varphi) \). On the other hand, suppose \( \xi \in G(\varphi) \), so that \( \varphi(P\xi) = \varphi(\xi) = 1 \). For any other torus calibration \( \varphi' \), \( \varphi'(P\xi) = \varphi'(\xi) \leq 1 \). It follows by elementary convex geometry from the characterization (*) of torus calibrations as

\[
\left\{\varphi \in T_S^* : \max_{\xi \in T} \langle \xi, \varphi \rangle = 1\right\}
\]

that \( P\xi \) is a convex combination of \( G_T(\varphi) \).
4. Definitions

We call $A \subset G(m, \mathbb{R}^n)$ a CP$^1$ if for some orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$ and the complex structure $Je_1 = e_3$, $Je_2 = e_4$ on $\mathbb{R}^4 = \text{span}\{e_1, e_2, e_3, e_4\}$.

$$A = \{\text{the complex lines in } \mathbb{R}^4\} \wedge e_5 \wedge \cdots \wedge e_{m+2}.$$  

Let $B \subset G(m, \mathbb{R}^n)$. We define the CP$^1$-closure $C(B)$ of $B$ as the smallest subset of $G(m, \mathbb{R}^n)$ containing $B$ such that whenever two points of a CP$^1$ lie in $C(B)$, the whole CP$^1$ lies in $C(B)$.

**Proposition 5** ([HM, Corollary 4.7]). Let $G(\varphi)$ be a face of $G(m, \mathbb{R}^n)$. If two points of a CP$^1$ lie in $G(\varphi)$, then the whole CP$^1$ lies in $G(\varphi)$.

**Theorem 6.** Let $\varphi \in \bigwedge^m \mathbb{R}^{2m^*}$ be a torus calibration. Then the face $G(\varphi)$ is the CP$^1$-closure of the torus face $G_T(\varphi)$:

$$G(\varphi) = C(T_T(\varphi)).$$

**Remarks.** This theorem subsumes both the Torus Lemma, which just says that $G_T(\varphi) \neq \emptyset$, and our new observation that the entire face of a torus calibration is determined by its torus face (cf. §3).

**Proof.** By Proposition 5, $G(\varphi) \supset C(G_T(\varphi))$. We prove the opposite inclusion by induction on $m$. The result is trivial for $m = 1$. Suppose $\varphi \in \bigotimes_{j=1}^{m+1} (\bigwedge^1 \mathbb{R}^{2^j}) \subset (\bigwedge^1 \mathbb{R}^{2^j}) \otimes (\bigwedge^m \mathbb{R}^{2m^*})$. Let $\xi \in G(\varphi)$. It is not hard to show ([HL, Lemma II.7.5]) that there are orthonormal bases $e_1, e_2$ for $\mathbb{R}^2$ and $f_1, \ldots, f_{2m}$ for $\mathbb{R}^{2m}$ and angles $\theta_1, \theta_2 \in [0, \pi/2]$ such that $\xi$ takes the form

$$\xi = (\cos \theta_1 e_1 + \sin \theta_1 f_1) \wedge (\cos \theta_2 e_2 + \sin \theta_2 f_2) \wedge f_3 \wedge \cdots \wedge f_{m+1}.$$  

Since $\varphi \in \bigwedge^1 \mathbb{R}^{2^*} \otimes \bigwedge^m \mathbb{R}^{2m^*}$,

$$\varphi(\xi) = a \cos_1 \sin \theta_2 + b \sin_1 \cos \theta_2 \leq \sqrt{a^2 \cos^2 \theta_1 + b^2 \sin^2 \theta_1} \leq \max\{|a|, |b|\} \leq 1,$$

where

$$a = \langle e_1 \wedge f_2 \wedge \cdots \wedge f_{m+1}, \varphi \rangle,$$

$$b = \langle f_1 \wedge e_2 \wedge \cdots \wedge f_{m+1}, \varphi \rangle.$$

Hence, equality holds. Unless $a = b = 1$, it follows that $\{\theta_1, \theta_2\} = \{0, \pi/2\}$ and $\xi$ has a factor $e_1$ or $e_2$, say $e_1$. Thus $\xi = e_1 \wedge \zeta$, for some $\zeta \in G(m, \mathbb{R}^{2m})$.

Since $e_1 \perp \varphi \in \bigotimes_{j=1}^{m} \bigwedge^1 \mathbb{R}^{2^j}$ and $\langle \zeta, e_1 \perp \varphi \rangle = \pm \langle e_1 \wedge \zeta, \varphi \rangle = \pm 1$, by induction $\pm \zeta$ lies in $C(G_T(e_1 \perp \varphi))$. Consequently $\xi = e_1 \wedge \zeta$ belongs to

$$\pm e_1 \wedge C(G_T(e_1 \perp \varphi)) \subseteq \pm C(e_1 \wedge G_T(e_1 \perp \varphi)) \subseteq C(G_T(\varphi))$$

as desired.
If on the other hand \( a = b = 1 \), then \( \theta_2 = \pi/2 - \theta_1 \). Also, \( e_1 \wedge f_2 \wedge \cdots \wedge f_{m+1} \) and \( f_1 \wedge e_2 \wedge \cdots \wedge f_{m+1} \) both belong to \( G(\varphi) \). As in the previous case, by induction both belong to \( C(G_T(\varphi)) \). In addition for the complex structure \( J e_1 = f_2, \ J f_1 = e_2 \), both belong to the \( \mathbb{C}P^1 \)

\[
A = \{ \text{complex lines in span } \{e_1, f_2, f_1, e_2\} \} \wedge f_3 \wedge \cdots \wedge f_{m+1}. 
\]

Since \( \theta_2 = \pi/2 - \theta_1 \), \( \xi \) also belongs to \( A \). Therefore \( \xi \in C(G_T(\varphi)) \), as desired.

References


[H2] ———, *Spinors and calibrations*, manuscript.


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