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UNSTABLE v_1-PERIODIC HOMOTOPY GROUPS OF A MOORE SPACE

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ABSTRACT. The mod p v_1-periodic homotopy groups of a space X are defined by considering the homotopy classes of maps of a Moore space into X and then inverting the Adams self-map of a Moore space. In this paper the mod p v_1-periodic homotopy groups of a Moore space are computed by using the Cohen-Moore-Neisendorfer splitting of the space of loops on a Moore space. The Adams map is shown to be compatible with this splitting and it is proved that the homomorphism of v_1-periodic homotopy groups induced by the Adams map is an isomorphism.

1. INTRODUCTION

Let X be a simply connected, p local space for some odd prime p. The degree p map of a sphere $S^n \xrightarrow{p} S^n$ acts on $\pi_*(X)$ and the rational homotopy groups of X are obtained by inverting this action. By analogy with this, the mod p v_1-periodic homotopy groups of a two-connected space X are defined as follows. Let $M^n = S^{n-1} \cup_p e^n$ be a mod p Moore space, and define the mod p homotopy groups of X by $\pi_n(X;\mathbf{Z}/p) = [M^n, X]$, $n \geq 3$. Let v_1 denote the Adams map $v_1\colon M^{n+q} \to M^n$, where $q = 2p-2$. The Adams map is constructed stably in [1], and in [10] it is shown that it can be constructed for $n \geq 3$. The basic feature of this map is that it induces an isomorphism in K-theory, hence all composites of v_1 with itself are essential. The groups $\pi_*(X;\mathbf{Z}/p)$ form a graded $\mathbf{Z}/p$ vector space, and a module over the ring $\mathbf{Z}/p[v_1]$ by defining αv_1 to be the composite $\alpha \circ v_1$, for $\alpha\colon M^n \to X$. Let K denote the graded field $\mathbf{Z}/p[v_1, v_1^{-1}]$. Define the mod p v_1-periodic homotopy groups of X by

$$v_1^{-1}\pi_*(X;\mathbf{Z}/p) \equiv \pi_*(X;\mathbf{Z}/p) \otimes_{\mathbf{Z}/p[v_1]} K.$$

A similar definition can be made for $p = 2$; however, in this paper we treat only the case of p odd.

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The first theorem of this paper is an explicit calculation of $v_1^{-1}\pi_*(X;\mathbf{Z}/p)$ when X is a mod p Moore space. The case $X = S^{2n+1}$ is the main topic of [17], and the method used there is based on the analogous method for $p = 2$ carried out by M. Mahowald in [14]; see also [15]. To analyze the v_1-periodic homotopy groups of M^n we follow a procedure suggested by Fred Cohen. In [6], [7], and [8] it is proved that ΩM^n splits as a product of certain spaces, and these factors in turn fit into fibrations involving spheres. The resulting long exact sequences of v_1-periodic homotopy groups yield the result.

Our second theorem asserts that the homomorphism of v_1-periodic homotopy groups induced by the Adams map is an isomorphism. This is not obvious, since the action of v_1 on $\pi_*(X;\mathbf{Z}/p)$ is obtained by composing on the left, whereas the homomorphism induced by v_1 is obtained by composing on the right. This theorem is analogous to analyzing the homomorphism of $\pi_*(S^n)$ induced by the degree k map. From another point of view, it is a special case of a conjecture that under suitable hypotheses, a map which induces an isomorphism in K-theory induces an isomorphism in $v_1^{-1}\pi_*\ (\ ;\mathbf{Z}/p)$. This is known to be true stably [4].

In order to state these results more precisely, we establish some notation. Fix an odd prime p and assume that all spaces are localized at p. Recall that K is the graded field $\mathbf{Z}/p[v_1, v_1^{-1}]$, where the degree of v_1 is q. Then for any 2-connected space X, $v_1^{-1}\pi_*(X;\mathbf{Z}/p)$ is a graded vector space over K. Denote $v_1^{-1}\pi_*(S^{2n+1};\mathbf{Z}/p)$ by $\mathscr{U}_{2n+1}$. It is proved in [17] that $\mathscr{U}_{2n+1}$ is four-dimensional over K with basis elements in dimensions $2n+1$, $2n+q$, $2np+q$, and $2np+2q-1$. Let $\mathscr{S}_{2n+1}$ denote the *stable* mod p v_1-periodic homotopy groups of S^{2n+1}. Then $\mathscr{S}_{2n+1} = v_1^{-1}\pi_*^S(S^{2n+1};\mathbf{Z}/p)$ is two-dimensional over K with basis elements in dimensions $2n+1$ and $2n+q$ by [16]. If W is a vector space over K, let $\Sigma^j W$ denote W shifted up in degree by j.

In [6] it is proved that there is a homotopy equivalence, for $n > 0$, $\Omega M^{2n+2} \simeq S^{2n+1}\{p\} \times \Omega(\bigvee_{j=0}^{\infty} M^{4n+2jn+3})$, where $S^{2n+1}\{p\}$ denotes the fiber of the degree p map on S^{2n+1}. Since S^{2n+1} is an H-space, the degree p map induces multiplication by p on $\pi_* S^{2n+1}$. Thus it is immediate that $v_1^{-1}\pi_*(\mathscr{S}^{2n+1}\{p\};\mathbf{Z}/p) = \mathscr{U}_{2n+1} \oplus \Sigma^{-1}\mathscr{U}_{2n+1}$. In [8] it is proved that there exists a space $T^{2n+1}\{p\}$, and a homotopy equivalence for $n > 1$, $\Omega M^{2n+1} \simeq T^{2n+1}\{p\} \times \Omega(\bigvee_\alpha M^{n_\alpha})$, where n_α is a certain set of indices satisfying $n_\alpha \geq 4n$, and only a finite number of indices occur in any dimension.

It follows from the above-mentioned splittings and the Hilton-Milnor theorem that ΩM^n, $n \geq 4$, splits as a weak product of spaces of the form $S^{2m+1}\{p\}$ and $T^{2m+1}\{p\}$. Thus in order to complete the computation of $v_1^{-1}\pi_*(M^n;\mathbf{Z}/p)$, it suffices to compute $v_1^{-1}\pi_*(T^{2m+1}\{p\};\mathbf{Z}/p)$, $m > 2$.

Theorem 1.1. *Let* $m > 2$. *Then* $v_1^{-1}\pi_*(T^{2m+1}\{p\};\mathbf{Z}/p) \cong \Sigma^{-2}\mathscr{S}_{2m+1} \oplus \Sigma^{-1}\mathscr{S}_{2m+1} \oplus (\bigoplus_{k\geq 1}^{\infty} \mathscr{U}_{2mp^k-1} \oplus \Sigma^{-1}\mathscr{U}_{2mp^k-1})$.

Concerning the homomorphism induced by v_1 we have

Theorem 1.2. *Let $n \geq 8$. The Adams map $v_1 \colon M^{n+q} \to M^n$ induces an isomorphism $(v_1)_* \colon v_1^{-1}\pi_*(M^{n+q};\mathbf{Z}/p) \xrightarrow{\cong} v_1^{-1}\pi_*(M^n;\mathbf{Z}/p)$.*

The author wishes to thank Fred Cohen for suggesting this problem, as well as Mark Mahowald for helpful discussion on this topic.

2. Preliminaries

In this section we will review the main results of [17] concerning $v_1^{-1}\pi_*(S^{2n+1};\mathbf{Z}/p) = \mathscr{U}_{2n+1}$ and we will summarize the splitting of ΩM^n from [6]–[8], as these are the main ingredients in proving Theorems 1.1 and 1.2.

Let B denote $(B\Sigma_p)_{(p)}$, the classifying space of the symmetric group Σ_p, localized at p. B^m will denote the m-skeleton. There is a Hopf-James map $\Omega_0^{2n+1}S^{2n+1} \xrightarrow{j} Q(B^{qn})$, also referred to as the Snaith map. See [9] for a definition of this map. The main theorem of [17] is that this map induces an isomorphism in $v_1^{-1}\pi_*(\ ;\mathbf{Z}/p)$. The v_1-periodic homotopy groups of QB^{qn} are the *stable* v_1-periodic homotopy groups $\Sigma^\infty B^{qn}$, which in turn are the same as $v_1^{-1}J_*(B^{qn};\mathbf{Z}/p)$, which can be explicitly computed. Here $J_*(\ )$ denotes homology based on the spectrum J, which is the fiber of a certain Adams operation $\theta \colon bu \to \Sigma^q bu$ in connective K-theory.

$\mathscr{U}_{2n+1}$ is generated over K by four classes in $\pi_*(S^{2n+1};\mathbf{Z}/p)$. The first, $\iota_{2n+1} \colon M^{2n+1} \to S^{2n+1}$, is the pinch map. The second, $\omega_{2n+1} \colon M^{2n+q} \to S^{2n+1}$, is the composite $M^{2n+q} \xrightarrow{\iota} S^{2n+q} \xrightarrow{\alpha_1} S^{2n+1}$, where α_1 is the generator of $\pi_{2n+q}(S^{2n+1}) \cong \mathbf{Z}/p$. The classes ι_{2n+1} and ω_{2n+1} are stable and generate $\mathscr{S}_{2n+1} = v_1^{-1}\pi_*^S(S^{2n+1};\mathbf{Z}/p)$. The other two classes are unstable, and can be described as follows. There are fibrations,

$$\begin{array}{ccc} W(n+1) & \xrightarrow{j} & \Omega^3 S^{2(n+1)p+1} \xrightarrow{d_1} \Omega S^{2(n+1)p-1} \\ \| & & \\ W(n+1) & \xrightarrow{i} & S^{2n+1} \xrightarrow{E_2} \Omega^2 S^{2(n+1)+1}, \end{array} \tag{2.1, 2.2}$$

where the lower fibration defines $W(n+1)$ as the fiber of the double-suspension map and the upper fibration comes from the EHP sequence. Since d_1 is degree p on the bottom cell, $(d_1)_*(\iota_{2(n+1)p+1}) = (d_1)_*(\omega_{2(n+1)p+1}) = 0$, and ι and ω pull back to classes in $\pi_*(W(n+1);\mathbf{Z}/p)$. $i_*j_*^{-1}(\iota_{2(n+1)p+1})$ and $i_*j_*^{-1}(\omega_{2(n+1)p+1})$ are the two unstable generators of $\mathscr{U}_{2n+1}$.

The above facts are proved by considering a map constructed by Cohen [5], from $W(n)$ to $Q(M^{2np-2})$, and by using an unstable Adams spectral sequence argument to show that this map induces an isomorphism in $v_1^{-1}\pi_*(\ ;\mathbf{Z}/p)$. The statements about S^{2n+1} follow by induction on n, using (2.2).

Now we summarize the splitting of ΩM^n, $n \geq 4$; see [6]–[8] for details. Let $x\colon M^k \to \Omega M^n$, $y\colon M^m \to \Omega M^n$ represent mod p homotopy classes. Since ΩM^n is a grouplike space, we can form Samelson products $[x, y]\colon M^k \wedge M^m \to \Omega M^n$. Since $M^k \wedge M^m$ splits as a wedge $M^{k+m} \vee M^{k+m-1}$ we can compose the inclusion of the "top" summand with the product to obtain $M^{k+m} \xrightarrow{[x,y]\circ i} \Omega M^n$, which will also be denoted by $[x, y]$ or $ad(x)(y)$. Thus $ad^r(x)(y)$ denotes $\underbrace{[x \ldots, [x, [x}_{r \text{ times}}, y]] \cdots]$.

Now let $v\colon M^{2n+1} \to \Omega M^{2n+2}$, $n \geq 1$, be adjoint to the identity. Let $u\colon M^{2n} \to \Omega M^{2n+2}$ be the Bockstein of v. We have mod p homotopy classes $ad^k(u)([v, v])\colon M^{4n+2+2kn} \to \Omega M^{2n+2}$ for $k \geq 0$. Taking the wedge of these we get $\bigvee_{k\geq 0}^{\infty} M^{4n+2+2kn} \to \Omega M^{2n+2}$. In [6] it is proved that the fiber of the adjoint map $\bigvee_{k\geq 0}^{\infty} M^{4n+3+2kn} \to M^{2n+2}$ is $S^{2n+1}\{p\}$ and the fibration

$$\Omega\left(\bigvee M^{4n+3+2nk}\right) \to \Omega M^{2n+2} \to S^{2n+1}\{p\}$$

is homotopically equivalent to a trivial fibration (i.e., a product). A splitting map is obtained from the diagram

$$\begin{tikzcd}
S^{2n+1}\{p\} \arrow[r] \arrow[d] & \Omega M^{2n+2} \arrow[d] \\
S^{2n+1} \arrow[r] \arrow[d, "p"] & * \arrow[d] \\
S^{2n+1} \arrow[r] & M^{2n+2}.
\end{tikzcd} \tag{2.3}$$

It is also necessary to consider ΩM^{2n+1}, $n > 1$. The first factor to split off is *not* a fiber of a degree p map on a sphere. Consider the following diagram:

$$\begin{tikzcd}
E^{2n+1} \arrow[r] \arrow[d] & M^{2n+1} \arrow[r] \arrow[d, equal] & S^{2n+1}\{p\} \arrow[d] \\
F^{2n+1} \arrow[r] \arrow[d] & M^{2n+1} \arrow[r, "q"] \arrow[d] & S^{2n+1} \arrow[d, "p"] \\
\Omega S^{2n+1} \arrow[r] & * \arrow[r] & S^{2n+1}
\end{tikzcd} \tag{2.4}$$

in which all rows and columns are fiber sequences, and the second and first rows define F^{2n+1} and E^{2n+1}. The map q pinches the bottom cell of M^{2n+1} to a point.

In [6] a certain set of mod p homotopy classes $M^{n_\alpha} \to F^{2n+1}$ is constructed. We will not record here the precise formula for n_α but only mention that these

classes represent relative Samelson products and the dimensions n_α satisfy (i) $n_\alpha \geq 4n$ and (ii) there are only finitely many n_α equal to any given dimension. In [6] it is shown that the fiber of the map $\bigvee_\alpha M^{n_\alpha} \to F^{2n+1}$ is homotopically equivalent to

$$S^{2n-1} \times \prod_{k\geq 1} S^{2np^k-1}\{p^2\}. \tag{2.5}$$

The above homotopy classes lift to E^{2n+1}, and in [7] it is proved that the fiber of $\bigvee_\alpha M^{n_\alpha} \to E^{2n+1}$ is

$$W(n) \times \prod_{k\geq 1} S^{2np^k-1}\{p^2\}. \tag{2.6}$$

Define $T^{2n+1}\{p\}$ to the fiber of the composite

$$\bigvee_\alpha M^{n_\alpha} \to F^{2n+1} \to M^{2n+1}.$$

In [8] it is shown that the fibration

$$\Omega\left(\bigvee_\alpha M^{n_\alpha}\right) \to \Omega M^{2n+1} \to T^{2n+1}\{p\}$$

is homotopically equivalent to a trivial fibration.

3. Proof of Theorem 1.1

Putting together (2.4), (2.5), and (2.6) we obtain a commutative diagram

$$\begin{array}{ccccc} \Omega^3 S^{2n+1} & \to & * & \to & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ W(n) \times \prod_{k\geq 1} S^{2np^k-1}\{p^2\} & \to & T^{2n+1}\{p\} & \to & \Omega S^{2n+1}\{p\} \\ \downarrow & & \| & & \downarrow \\ S^{2n-1} \times \prod_{k\geq 1} S^{2np^k-1}\{p^2\} & \to & T^{2n+1}\{p\} & \to & \Omega S^{2n+1} \end{array} \tag{3.1}$$

in which the rows and columns are fiber sequences, the right-hand column is the fiber sequence which defines $S^{2n+1}\{p\}$, and the left-hand column is the product of the fiber sequence which defines $W(n)$ and the fibration

$$* \to \prod S^{2np^k-1}\{p^2\} \xrightarrow{=} \prod S^{2np^k-1}\{p^2\}.$$

To prove Theorem 1.1, it suffices to compute the boundary homomorphism δ in the long exact sequence in $v_1^{-1}\pi_*(\ ;\mathbf{Z}/p)$ for the middle row.

Recall that $v_1^{-1}\pi_*(\Omega S^{2n+1}\{p\};\mathbf{Z}/p) \cong \Sigma^{-2}\mathscr{U}_{2n+1} \oplus \Sigma^{-1}\mathscr{U}_{2n+1}$. The two stable classes $\Sigma^{-2}\iota_{2n+1}$ and $\Sigma^{-2}\omega_{2n+1}$ in the first summand, corresponding to the upper-right-hand sphere of 3.1, are in the kernel of δ since they map to 0 in $\pi_*(W(n);\mathbf{Z}/p)$. The two unstable classes in the first summand are not in the kernel of δ, since they map nontrivially to $\pi_*(W(n);\mathbf{Z}/p)$.

To evaluate δ on the second summand, it suffices to consider the boundary map δ' for the third row. The boundary map δ', followed by projection onto

the first factor, is represented by a map $\varphi_{2n+1}\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ which is shown in [6], [7] to be degree p on the bottom cell. In a moment we will prove:

Proposition 3.2. *Assume $n > 2$. Let $\varphi_{2n+1}\colon \Omega^2 \mathscr{S}^{2n+1} \to S^{2n-1}$ be any map which is degree p on the bottom cell, i.e., such that*

$$\begin{array}{ccc} \Omega^2 \mathscr{S}^{2n+1} & \xrightarrow{\varphi_{2n+1}} & S^{2n-1} \\ \uparrow E^2 & \nearrow p & \\ S^{2n-1} & & \end{array}$$

commutes. Then the two stable elements $\Sigma^{-2}\iota_{2n+1}$ and $\Sigma^{-2}\omega_{2n+1}$ in $v_1^{-1}\pi_*(\Omega^2 S^{2n+1};\mathbf{Z}/p)$ are in the kernel of $(\varphi_{2n+1})_*$, and the two unstable classes are mapped onto the two unstable classes in $v_1^{-1}\pi_*(S^{2n-1};\mathbf{Z}/p)$.

It follows from Proposition 3.2 that the unstable classes in $\Sigma^{-1}\mathscr{U}_{2n+1}$ are not in the kernel of δ'. The stable classes are mapped by δ' to elements which are 0 in the first component by Proposition 3.2. They have the remaining components equal to 0 because the connectivity of $S^{2np^k-1}\{p^2\}$ is greater than the dimension of $\Sigma^{-1}\iota_{2n+1}$ and $\Sigma^{-1}\omega_{2n+1}$. This completes the proof of Theorem 1.1. □

Proof of Proposition 3.2. Since the degree p map induces 0 in the $\pi_*(\ ;\mathbf{Z}/p)$, and the class $\Sigma^{-2}\iota_{2n+1}$ and $\Sigma^{-2}\omega_{2n+1}$ are in the image of double suspension, the first assertion follows immediately. For the statement concerning the unstable classes, consider the following commutative diagram:

$$\begin{array}{ccccc} QB^{q(n-1)} & & \xrightarrow{\quad p \quad} & & QB^{q(n-1)} \\ \uparrow j & & & & \uparrow j \\ \Omega^{2n-1}S^{2n-1} & \xrightarrow{\Omega^{2n-1}E^2} & \Omega^{2n+1}S^{2n+1} & \xrightarrow{\Omega^{2n-1}\varphi_{2n+1}} & \Omega^{2n-1}S^{2n-1} \\ \uparrow \lambda & & \uparrow \lambda & & \\ B^{q(n-1)} & \xrightarrow{\ i\ } & B^{qn} & & \end{array}$$

The maps λ are those described in [11].

The generators of $v_1^{-1}\pi_*(\Omega^{2n+1}S^{2n+1};\mathbf{Z}/p)$ are constructed in $\pi_*(B^{qn})$ and detected in $\pi_*(QB^{qn}) = \pi_*^S(B^{qn})$. Thus in order to evaluate $(\varphi_{2n+1})_*$ on the classes in $v_1^{-1}\pi_*(S^{2n+1};\mathbf{Z}/p)$ it suffices to consider any *stable* map $B^{qn} \xrightarrow{\varphi} B^{q(n-1)}$ which is the degree p map when restricted to $B^{q(n-1)}$. Given such a map there is an extension ψ:

$$\begin{array}{ccc} B^{qn}_{2q-1} & & \\ \uparrow r & \searrow \psi & \\ B^{qn} & \xrightarrow{\varphi} & B^{q(n-1)} \end{array}$$

where B^{qn}_{2q-1} is the cofiber of the inclusion $M^q \hookrightarrow B^{qn}$. The map ψ is easily seen to induce an isomorphism in K-theory, hence in stable mod v_1-periodic

homotopy by Theorem 4.11 of [4]. The classes in $v_1^{-1}\pi_*^S(B^{qn})$ which correspond to the unstable generators of $\mathscr{U}_{2n+1}$ are mapped nontrivially by r_*, hence by φ_*, which implies the result. □

4. Proof of Theorem 1.2

A classical method for analyzing the homomorphism of $\pi_* S^n$ induced by the degree k map $S^n \xrightarrow{k} S^n$ is to write k as a composite:

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \vee \cdots \vee S^n \xrightarrow{\text{fold}} S^n,$$

then loop once and apply the Hilton-Milnor theorem to the space $\Omega(\bigvee S^n)$:

$$\begin{array}{ccccc} \Omega S^n & \to & \Omega(\bigvee S^n) & \to & \Omega S^n \\ \searrow & & \downarrow \cong & & \nearrow \\ & & \prod_j S^{n_j}. & & \end{array}$$

In a somewhat analogous fashion, we analyze $(v_1)_*$ by looping v_1, then applying the splitting of ΩM^n. We first need to establish that the splitting of ΩM^n behaves well with respect to the Adams map. Actually, the Adams map does *not* commute with the Bockstein, as a simple check of the Steenrod operations shows; and for this reason we need to consider the pth power of the Adams map. Let u, v be as in §2, and let $f_k\colon M^{4n+3+2kn} \to M^{2n+2}$ be the adjoint of $ad^k(u)([v,v])$. Let $m = n + p(p-1)$.

Proposition 4.1. *There exists a commutative diagram, for* $n \geq 2$.

$$\begin{array}{ccc} \bigvee_{k\geq 0} M^{4m+3+2km} & \xrightarrow{\vee f_k} & M^{2m+2} \\ \downarrow \vee v_1^{(2+k)p} & & \downarrow v_1^p \\ \bigvee_{k\geq 0} M^{4n+3+2kn} & \xrightarrow{\vee f_k} & M^{2n+2}. \end{array}$$

This will be proved at the end of the section.

Similarly, for an odd-dimensional Moore space, let $\{n_\alpha\}$ be as in §2, and denote by $g_\alpha\colon M^{n_\alpha} \to M^{2n+1}$ the Samelson product referred to there.

Proposition 4.2. *Let* $n \geq 3$, *and* $m = n + p(p-1)$. *Then there exists a commutative diagram*

$$\begin{array}{ccc} \bigvee_\alpha M^{m_\alpha} & \xrightarrow{\vee g_\alpha} & M^{2m+1} \\ \downarrow v_1^{pk_\alpha} & & \downarrow v_1^p \\ \bigvee_\alpha M^{n_\alpha} & \xrightarrow{\vee g_\alpha} & M^{2n+1} \end{array}$$

for certain indices $\{k_\alpha\}$.

Now consider the induced map of homotopy fibers from diagrams 4.1 and 4.2 respectively.

$$\begin{array}{lcl} \alpha\colon S^{2m+1}\{p\} & \to & S^{2n+1}\{p\} \\ \beta\colon T^{2m+1}\{p\} & \to & T^{2n+1}\{p\}. \end{array}$$

Since ΩM^n splits as a weak product of spaces of the form $S^k\{p\}$ and $T^k\{p\}$, Theorem 1.2 will follow from repeated application of the following:

Proposition 4.3. *The maps α and β defined above induce isomorphisms in $v_1^{-1}\pi_*(\ ;\mathbf{Z}/p)$.*

To prove this it suffices to consider $\Omega^2\alpha$ and $\Omega^2\beta$. Proposition 4.3 will be deduced from the fact that $\Omega^2\alpha$ and $\Omega^2\beta$ induce isomorphisms in mod p K-theory, and from the fact that the v_1-periodic homotopy groups of $\Omega^2 S^k\{p\}$ and $\Omega^2 T^k\{p\}$ are detected stably, when k is odd.

In what follows, let $K_*(\ )$ denote mod p K-theory and let K denote the representing spectrum.

Lemma 4.4. *Let $F_i \to E_i \to B_i$, $i = 0,1$ be a principal homotopy fiber sequence. Let*

$$\begin{array}{ccccc} F_0 & \longrightarrow & E_0 & \longrightarrow & B_0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ F_1 & \longrightarrow & E_1 & \longrightarrow & B_1 \end{array}$$

be a map of principal fiber sequences. If f_1 and f_2 induce an isomorphism in $K_*(\ )$, then so does f_3.

Proof. We have a Steenrod-Rothenberg type spectral sequence with

$$E_2 = \mathrm{Tor}_{K_*(F_i)}(K_*(E_i), \mathbf{Z}/p)$$

abutting to $K_*(B_i)$ [2] [18]. The maps f_i induce a map of spectral sequences and the hypothesis implies that this is an isomorphism on E_2, hence on E_∞. □

Corollary 4.5. *The maps $\Omega^2\alpha$ and $\Omega^2\beta$ defined above induce isomorphisms in $K_*(\ )$.*

Proof. For $\Omega^2\alpha$, apply Lemma 4.4 to the diagram

$$\begin{array}{ccccc} \Omega^3(\vee M^{4m+3+2km}) & \longrightarrow & \Omega^3 M^{2m+2} & \longrightarrow & \Omega^2 S^{2m+1}\{p\} \\ \downarrow & & \downarrow{\scriptstyle \Omega^3 v_1^p} & & \downarrow{\scriptstyle \Omega^2\alpha} \\ \Omega^3(\vee M^{4n+3+2kn}) & \longrightarrow & \Omega^3 M^{2n+2} & \longrightarrow & \Omega^2 S^{2n+1}\{p\} \end{array}$$

and note that by an argument similar to the proof of Proposition 2.4 of [13], the functor $\Omega^3\Sigma^3(\ )$ applied to a $K_*(\ )$ isomorphism gives a $K_*(\ )$ isomorphism. We are using the fact that $2n+2 \geq 6$. Similar argument holds for the map β. □

We must investigate under what circumstances maps inducing $K_*(\ )$ isomorphisms also induce $v_1^{-1}\pi_*(\ ;\mathbf{Z}/p)$ isomorphisms. The simplest situation to handle can be described as follows.

Let $\mathscr{J}$ denote the nonconnective mod p J spectrum, i.e, $\mathscr{J}$ is the fiber of a certain Adams operation $\theta: K \to K$, as in [4, §4]. There is a unit map $S^0 \xrightarrow{i} \mathscr{J}$, which induces, for any space X, a map

$$X \xrightarrow{h} \Omega^\infty(\mathscr{J} \wedge \Sigma^\infty X).$$

The induced homomorphism $\pi_* X \xrightarrow{h_*} \mathscr{J}_*(X)$ is the mod p $\mathscr{J}$-homology Hurewicz homomorphism.

Proposition 4.6. *Let $f: X \to Y$ be a map of 2 connective spaces. Suppose f is injective in $\mathscr{J}_*(\ )$, and suppose $\gamma: M^n \to X$ is a v_1-periodic homotopy class which is not in the kernel of the $\mathscr{J}_*$ Hurewicz map. Then $f_*(\gamma)$ is essential and v_1-periodic in $\pi_*(Y; \mathbf{Z}/p)$.*

Proof. The proof is immediate from the naturality of the Hurewicz homomorphism and from the fact that all the (stable) mod p homotopy groups of the spectrum $\mathscr{J} \wedge \Sigma^\infty Y$ are v_1-periodic. □

Corollary 4.7. *If $f: X \to Y$ induces an isomorphism in $K_*(\ )$, and the kernel of $h_*: \pi_*(X; \mathbf{Z}/p) \to \mathscr{J}_*(X; \mathbf{Z}/p)$ is all v_1-torsion, then f induces an injection in $v_1^{-1}\pi_*(\ ; \mathbf{Z}/p)$.*

Proof. The proof is immediate from Proposition 4.6 and from the fact that a $K_*(\ )$ isomorphism will also be a $\mathscr{J}_*(\ )$ isomorphism. □

Proof of Proposition 4.3. For the map α, it suffices to show that $\Omega^2\alpha$ is injective in v_1-periodic homotopy since $v_1^{-1}\pi_*(\Omega^2 S^{2n+1}\{p\}; \mathbf{Z}/p)$ is eight-dimensional over $\mathbf{Z}/p[v_1, v_1^{-1}]$. This in turn follows from Corollaries 4.5 and 4.7 once we show that none of the eight generators of $v_1^{-1}\pi_*(\Omega^2 S^{2n+1}\{p\}; \mathbf{Z}/p)$ are in the kernel of h_*. To do this, it suffices to find, for each generator of $v_1^{-1}\pi_*(\Omega^2 S^{2n+1}\{p\}; \mathbf{Z}/p)$, a map $\Omega^2 S^{2n+1}\{p\} \to Q(M^k)$, where M^k is some Moore space, such that the generator maps to a stable v_1-periodic homotopy class in $\pi_*^S(M^k; \mathbf{Z}/p)$. In this case we will say that the generator is detected stably.

Consider first the four classes in $v_1^{-1}\pi_*(S^{2n+1}\{p\}; \mathbf{Z}/p) = \Sigma^{-1}\mathscr{U}_{2n+1} \oplus \mathscr{U}_{2n+1}$, which correspond to the two stable classes ι and ω in each summand. There is a diagram

$$M^{2n+1} \to S^{2n+1}\{p\} \to \Omega M^{2n+2} \to Q(M^{2n+1})$$

where the first map is inclusion of the bottom two cells, the second map is the splitting map of 2.3, and the third map is stabilization; and it is readily verified that the four generators of the stable group $v_1^{-1}\pi_*^S(M^{2n+1}; \mathbf{Z}/p)$ lift to classes in $v_1^{-1}\pi_*(M^{2n+1}; \mathbf{Z}/p)$, which map to the four classes in $v_1^{-1}\pi_*(S^{2n+1}\{p\}; \mathbf{Z}/p)$ in question. Looping twice yields the desired map $\Omega^2 S^{2n+1}\{p\} \to Q(M^{2n-1})$.

As for the other four generators, let $\varphi_{2n+1}: \Omega^2 S^{2n+1} \to S^{2n-1}$, $n \geq 2$, be the map constructed in [6] and referred to in §3. In [7] it is shown that φ_{2n+1}

satisfies the commutative diagram

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\varphi_{2n+1}} & S^{2n-1} \\ & \Omega^2 p \searrow & \downarrow E^2 \\ & & \Omega^2 S^{2n+1}. \end{array}$$

Thus we have a diagram

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\ \uparrow \Omega^2 p & & \uparrow E^2 \\ \Omega^2 S^{2n+1} & \xrightarrow{\varphi_{2n+1}} & S^{2n-1} \\ \uparrow & & \uparrow \\ \Omega^2 S^{2n+1}\{p\} & \xrightarrow{\nu} & W(n) \end{array} \tag{4.8}$$

where the vertical sequences are fiber sequences and ν is the map of fibers. It follows from (4.8) and Proposition 3.2 that ν maps the four classes under consideration in $v_1^{-1}\pi_*(\Omega^2 S^{2n+1}\{p\}; \mathbf{Z}/p)$ to the four v_1-periodic classes in $\pi_*(W(n); \mathbf{Z}/p)$, which are in turn detected by Cohen's map $W(n) \to Q(M^{2np-2})$. This completes the proof that none of the classes in $v_1^{-1}\pi_*(\Omega^2 S^{2n+1}\{p\}; \mathbf{Z}/p)$ are in the kernel of h_*.

It remains to prove that β induces an isomorphism in v_1-periodic homotopy. The map $\Omega M^{2m+1} \to \Omega S^{2m+1}\{p\}$ given by

$$\Omega M^{2m+1} \xrightarrow{\Omega\Sigma} \Omega^2 M^{2m+2} \to \Omega S^{2m+1}\{p\}$$

also factors as

$$\Omega M^{2m+1} \to T^{2m+1}\{p\} \to \Omega S^{2m+1}\{p\}.$$

It follows that the outside rectangle in the following diagram commutes:

$$\begin{array}{ccccc} \Omega M^{2m+1} & \longrightarrow & T^{2m+1}\{p\} & \longrightarrow & \Omega S^{2m+1}\{p\} \\ \downarrow \Omega v_1^p & & \downarrow \beta & & \downarrow \Omega\alpha \\ \Omega M^{2n+1} & \longrightarrow & T^{2n+1}\{p\} & \longrightarrow & \Omega S^{2n+1}\{p\}. \end{array}$$

The left-hand square commutes by definition of β. Since $T^{2m+1}\{p\}$ is a retract of ΩM^{2m+1}, it follows that the right-hand square also commutes. Let $V^{2m+1}\{p\}$ denote $W(m) \times \prod_{k\geq 1} S^{2mp^k-1}\{p^2\}$, recall the fibration which is the middle row of 3.1, and let $\gamma\colon V^{2m+1}\{p\} \to V^{2n+1}\{p\}$ be the map of fibers induced by the right-hand square. By Lemma 4.4, $\Omega^2\gamma$ induces an isomorphism in $K_*(\ )$.

For each k there is a fibration

$$S^{2mp^k-1}\{p\} \xrightarrow{i} S^{2mp^k-1}\{p^2\} \xrightarrow{j} S^{2mp^k-1}\{p\}.$$

Of the eight generators of $v_1^{-1}\pi_*(S^{2mp^k-1}\{p^2\}; \mathbf{Z}/p)$, four are in the image of i_* and four are not in the kernel of j_*. The latter four are detected stably after looping twice, as shown above; hence these four generators are mapped

nontrivially by $\Omega^2\gamma$. As for the generators in the image of i_*, first note that the map $S^{2mp^k-1}\{p^2\} \to T^{2m+1}\{p\}$ is constructed by composing $S^{2mp^k-1}\{p^2\} \to \Omega M^{2mp^k}(p^2)$ with a Samelson product $\Omega M^{2mp^k}(p^2) \to \Omega M^{2m+1}$ [6]–[8]. This Samelson product is an extension over the mod p^2 Moore space of a Samelson product $\Omega M^{2mp^k}(p) \to \Omega M^{2m+1}$; hence we have a diagram,

$$\begin{array}{ccccc} S^{2mp^k-1}\{p\} & \to & \Omega M^{2mp^k}(p) & \to & \Omega M^{2m+1} \\ & & \downarrow \Omega v_1^{j_k} & & \downarrow \Omega v_1^p \\ & & \Omega M^{2np^k}(p) & \to & \Omega M^{2n+1}, \end{array}$$

and the fact that these four generators are mapped nontrivially by $\Omega^2\gamma$ follows from our previous analysis of the map α.

Note that $v_1^{-1}\pi_*(V^{2m+1}\{p\};\mathbf{Z}/p)$ is not finite dimensional. However it is filtered by

$$F_t^m = \operatorname{im}(v_1^{-1}\pi_*(W(m) \times \prod_{k=1}^{t} S^{2mp^k-1}\{p^2\};\mathbf{Z}/p) \to v_1^{-1}\pi_*(V^{2m+1}\{p\};\mathbf{Z}/p))$$

and F_t^m is finite dimensional. We need a lemma. Recall that $m = n+p(p-1)$.

Lemma 4.9. *Given a map $f\colon S^{2mp^k-1}\{p^2\} \to S^{2np^j-1}\{p^2\}$ such that $j > k$, the induced homomorphism in v_1-periodic homotopy groups is zero.*

It follows from Lemma 4.9 that $(\Omega^2\gamma)_*$ is filtration preserving. Since F_t^m is finite dimensional, and $(\Omega^2\gamma)_*$ is injective, $(\Omega^2\gamma)_*|_{F_t^m}$ is an isomorphism. Since $v_1^{-1}\pi_*(V^{2m+1}\{p\};\mathbf{Z}/p) = \bigcup_t F_t^m$, we have that $(\Omega^2\gamma)_*$ is an isomorphism. Since γ_* and α_* are isomorphisms, it follows by The Five Lemma that B_* is also one.

Proof of Lemma 4.9. The quotient of $\pi_*(S^{2l+1}\{p^2\};\mathbf{Z}/p)$ by the subgroup of v_1-torsion elements has eight generators in dimensions $2l$, $2l+1$, $2l+q-1$, $2l+q$, $2lp+q-1$, $2lp+q$, $2lp+2q-2$, and $2lp+2q-1$, as in (2.2). These dimensions are minimal, in other words these generators are not divisible by v_1. The first four of these are stable in $S^{2l+1}\{p^2\}$, and the second four are unstable.

Consider the given map f. The first four stable generators of $\pi_*(S^{2mp^k-1}\{p^2\};\mathbf{Z}/p)/(v_1 - \text{torsion})$ are in the kernel of f_* because their dimension is less than the connectivity of $S^{2np^j-1}\{p^2\}$. The second four unstable generators are in dimensions which are less than the dimensions of the four unstable generators in $\pi_*(S^{2np^j-1}\{p^2\};\mathbf{Z}/p)$ so they cannot map to these unstable classes. Furthermore, an unstable homotopy class cannot map nontrivially to a stable homotopy class; hence the four unstable generators are in the kernel of f_*, proving the lemma. □

We conclude with the proof of Propositions 4.1 and 4.2. Assume x, $y \in \pi_*(\Omega M^n; \mathbf{Z}/p)$, x', $y' \in \pi_*(\Omega M^m; \mathbf{Z}/p)$ such that the following diagrams commute:

$$\begin{array}{ccccc} M^{k'} & \xrightarrow{x'} & \Omega M^m & M^{\ell'} & \xrightarrow{y'} \Omega M^m \\ \downarrow v_1^i & & \downarrow \Omega v_1^s & \downarrow v_1^j & \downarrow \Omega v_1^s \\ M^k & \xrightarrow{x} & \Omega M^n & M^{\ell} & \xrightarrow{y} \Omega M^n. \end{array} \tag{4.10}$$

Then we have a commutative diagram

$$\begin{array}{ccccc} M^{k'+\ell'} & \longrightarrow & M^{k'} \wedge M^{\ell'} & \xrightarrow{[x',y']} & \Omega M^m \\ \downarrow v_i^{i+j} & & \downarrow v_1^i \wedge v_1^j & & \downarrow \\ M^{k+\ell} & \longrightarrow & M^k \wedge M^{\ell} & \xrightarrow{[x,y]} & \Omega M^n. \end{array} \tag{4.11}$$

The commutativity of the right-hand square is just naturality of the Samelson product; and the commutativity of the left-hand square, in other words, the compatibility of the splitting of the $M^k \wedge M^{\ell}$ with the Adams map, is easy to check and was first proved in [12].

Now let $v\colon M^{n-1} \to \Omega M^n$ be as in §2, adjoint to the identity map. The diagram

$$\begin{array}{ccc} M^{n+q-1} & \xrightarrow{v} & \Omega M^{n+q} \\ \downarrow v_1 & & \downarrow \Omega v_1 \\ M^{n-1} & \xrightarrow{v} & \Omega M^n \end{array}$$

commutes since the map $X \to \Omega\Sigma(X)$ is natural.

A simple check of Steenrod operations shows that

$$\begin{array}{ccc} M^{n+q-2} & \xrightarrow{v_1} & M^{n-2} \\ \downarrow \beta & & \downarrow \beta \\ M^{n+q-1} & \xrightarrow{v_1} & M^{n-1} \end{array}$$

does *not* commute. However, if we replace v_1 by v_1^p, then $v_1^p \beta = \beta v_1^p$, as was proved in [12]. For this we need $n \geq 8$. Hence we have

$$\begin{array}{ccccc} M^{n+qp-2} & \xrightarrow{\beta} & M^{n+pq-1} & \xrightarrow{v} & \Omega M^{n+pq} \\ \downarrow v_1^p & & \downarrow v_1^p & & \downarrow \Omega v_1^p \\ M^{n-2} & \xrightarrow{\beta} & M^{n-1} & \xrightarrow{v} & \Omega M^n \end{array}$$

where the composite $v\beta$ is the map u.

It now follows that all iterated Samelson products of u and v are compatible with the Adams map, by induction on the length of the Samelson product. Since the maps f_k and g_α of Propositions 4.1 and 4.2 are of this form, the conclusion follows. □

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