A NEW EXAMPLE IN $K$-THEORY OF LOOPSPACES

TAHSIN GHAZAL

(Communicated by Frederick R. Cohen)

Abstract. The “Eilenberg-Moore” type spectral sequences which connect $K^*(\Omega X)$ and $K^*(X)$ have well-known bad properties, when, for example, $X = K(\mathbb{Z}/p, 2)$. This paper shows that the result can be as bad when $X$ is a finite complex.

The “Eilenberg-Moore” type spectral sequences which connect $K^*(\Omega X)$ and $K^*(X)$ have well-known bad properties, such as when $X = K(\mathbb{Z}/p, 2)$ [4]. The purpose of this note is to show that the spectral sequence properties can be just as bad when $X$ is a finite complex. Specifically, we prove

Theorem 1. For any odd prime $p$, there is a simply connected finite complex $X$ such that

(i) $\tilde{K}^*(X; \mathbb{Z}) = 0$ (integral $K$-theory)
(ii) $\tilde{K}^*(\Omega X; \mathbb{Z})$ contains a direct summand isomorphic to $\mathbb{Z}/p$.

It follows (informally) that there is no way of deducing $K^*(\Omega X; \mathbb{Z})$ from the limit of the Eilenberg-Moore spectral sequence with $E_2 = \text{Tor}_{K^*(X; \mathbb{Z})}(\mathbb{Z}, \mathbb{Z})$, even when $X$ is finite.

The example $X$ is a very obvious one; the cofibre of the Adams map [1] in a suitable dimension. We describe this space $X$ and prove that it has trivial $K$-theory in §1; in §2 we show that its loop space has nontrivial $K$-theory.

1. THE SPACE $X$

Let $p > 1$ be a prime and let $P^n(p)$ denote the Moore space $S^{n-1} U_p e^n$. In [1] J. F. Adams described stable maps

(1) $A: P^{n+2p-2}(p) \to P^n(p)$

($n$ large) such that $A^*$ is an isomorphism in $K$-theory. We define $X(n, p)$ to be the cofibre of $A$:

(2) $X(n, p) = P^n(p) U_A C(P^{n+2p-2}(p))$.

Received by the editors August 25, 1988 and, in revised form, March 1, 1989.
Key words and phrases. Adams map, $K$-theory.
Lemma 1. \( \tilde{K}^*(X(n,p);\mathbb{Z}) = 0. \)

Proof. The proof follows immediately from the cofibration exact sequence.

Now choose \( n \) so that \( \tilde{K}^*(\Omega X;\mathbb{Z}) \) is not 0. The construction is only possible when \( n \) has its minimum value \( n_0 \), since for \( n > n_0 \), \( X \) is a suspension \( \Sigma Y \), with \( \tilde{K}^*(Y;\mathbb{Z}) = 0 \). By the theory of the James construction [5], \( \Omega X = \Omega \Sigma Y \) also has trivial \( K \)-theory in this case.

Cohen and Neisendorfer [3] have computed the minimum value for \( n \) to be 3. We therefore set \( X = X(3,p) = P^3(p)U_A C(P^{2p+1}(p)) \). We have \( H^*(X;\mathbb{Z}) \) given by

\[
\begin{align*}
H^3(X;\mathbb{Z}) &= H^{2p+2}(X;\mathbb{Z}) = \mathbb{Z}/p, \\
H^j(X;\mathbb{Z}) &= 0 \quad \text{otherwise}.
\end{align*}
\]

2. The \( K \)-theory of \( \Omega X \)

Proving that the \( K \)-theory of \( \Omega X \) is nontrivial is simple. Using (3) and the transgression exact sequence of [6] gives

Lemma 2. \( H^1(\Omega X;\mathbb{Z}) = 0; \quad H^2(\Omega X;\mathbb{Z}) = \mathbb{Z}/p. \)

But \( H^2(\Omega X;\mathbb{Z}) \) is a direct summand in \( \tilde{K}^0(\Omega X;\mathbb{Z}) \), as is well known. (The mapping \( K(\mathbb{Z},2) = BU(1) \to BU \) is split by the determinant map.) This completes the proof of Theorem 1.

We can prove the analogue to Theorem 1 for \( K \)-theory mod \( p \) using the same space \( X \); a careful use of the universal coefficient theorem [2] will provide a \( \mathbb{Z}/p \) summand in \( K^0(\Omega X;\mathbb{Z}/p). \)

Of course the Adams maps are defined (in suitable degrees) if \( p \) is replaced by \( p' \):

\[
A_r : P^{n+2(p-1)p'-1}(p') \to P^n(p').
\]

If we know the exact level of desuspension of \( A_r \), we can expect a similar result for \( K \)-theory of the cofibre and its loopspace.

References


Mathematics Department, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.