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A NEW EXAMPLE IN K-THEORY OF LOOPSPACES

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ABSTRACT. The "Eilenberg-Moore" type spectral sequences which connect $K^*(\Omega X)$ and $K^*(X)$ have well-known bad properties, when, for example, $X = K(\mathbb{Z}/p, 2)$. This paper shows that the result can be as bad when X is a finite complex.

The "Eilenberg-Moore" type spectral sequences which connect $K^*(\Omega X)$ and $K^*(X)$ have well-known bad properties, such as when $X = K(\mathbb{Z}/p, 2)$ [4]. The purpose of this note is to show that the spectral sequence properties can be just as bad when X is a finite complex. Specifically, we prove

Theorem 1. *For any odd prime p, there is a simply connected finite complex X such that*

(i) $\widetilde{K}^*(X;\mathbb{Z}) = 0$ *(integral K-theory)*
(ii) $\widetilde{K}^*(\Omega X;\mathbb{Z})$ *contains a direct summand isomorphic to $\mathbb{Z}/p$.*

It follows (informally) that there is no way of deducing $K^*(\Omega X;\mathbb{Z})$ from the limit of the Eilenberg-Moore spectral sequence with $E_2 = \mathrm{Tor}_{K^*(X;\mathbb{Z})}(\mathbb{Z}, \mathbb{Z})$, even when X is finite.

The example X is a very obvious one; the cofibre of the Adams map [1] in a suitable dimension. We describe this space X and prove that it has trivial K-theory in §1; in §2 we show that its loopspace has nontrivial K-theory.

1. THE SPACE X

Let $p > 2$ be a prime and let $P^n(p)$ denote the Moore space $S^{n-1}U_p e^n$. In [1] J. F. Adams described stable maps

$$A\colon P^{n+2p-2}(p) \to P^n(p) \tag{1}$$

(n large) such that A^* is an isomorphism in K-theory. We define $X(n,p)$ to be the cofibre of A:

$$X(n,p) = P^n(p)U_A C(P^{n+2p-2}(p)). \tag{2}$$

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Lemma 1. $\widetilde{K}^*(X(n,p);\mathbb{Z}) = 0$.

Proof. The proof follows immediately from the cofibration exact sequence.

Now choose n so that $\widetilde{K}^*(\Omega X;\mathbb{Z})$ is not 0. The construction is only possible when n has its minimum value n_0, since for $n > n_0$, X is a suspension ΣY, with $\widetilde{K}^*(Y;\mathbb{Z}) = 0$. By the theory of the James construction [5], $\Omega X = \Omega\Sigma Y$ also has trivial K-theory in this case.

Cohen and Neisendorfer [3] have computed the minimum value for n to be 3. We therefore set $X = X(3,p) = P^3(p)U_A C(P^{2p+1}(p))$. We have $H^*(X;\mathbb{Z})$ given by

$$\begin{aligned} &H^3(X;\mathbb{Z}) = H^{2p+2}(X;\mathbb{Z}) = \mathbb{Z}/p, \\ &H^j(X;\mathbb{Z}) = 0 \quad \text{otherwise.} \end{aligned} \tag{3}$$

2. The K-theory of ΩX

Proving that the K-theory of ΩX is nontrivial is simple. Using (3) and the transgression exact sequence of [6] gives

Lemma 2. $H^1(\Omega X;\mathbb{Z}) = 0$; $H^2(\Omega X;\mathbb{Z}) = \mathbb{Z}/p$.

But $H^2(\Omega X;\mathbb{Z})$ is a direct summand in $\widetilde{K}^0(\Omega X;\mathbb{Z})$, as is well known. (The mapping $K(\mathbb{Z},2) = BU(1) \to BU$ is split by the determinant map.) This completes the proof of Theorem 1.

We can prove the analogue to Theorem 1 for K-theory mod p using the same space X; a careful use of the universal coefficient theorem [2] will provide a $\mathbb{Z}/p$ summand in $K^0(\Omega X;\mathbb{Z}/p)$.

Of course the Adams maps are defined (in suitable degrees) if p is replaced by p^r:

$$A_r\colon P^{n+2(p-1)p^{r-1}}(p^r) \to P^n(p^r).$$

If we know the exact level of desuspension of A_r, we can expect a similar result for K-theory of the cofibre and its loopspace.

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