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A DIAMETER PINCHING SPHERE THEOREM FOR POSITIVE RICCI CURVATURE

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ABSTRACT. In this note we generalize Shiohama's volume pinching sphere theorem to a diameter pinching sphere theorem for positive Ricci curvature.

1. INTRODUCTION

In this paper a manifold M always means a complete connected Riemannian manifold of dimension n and $v(M)$ will denote the volume of M, $d(M)$ the diameter of M, K_M the sectional curvature of M and Ric_M the Ricci curvature of M.

The sphere theorem due to Klingenberg [K] says that if M is a complete, simply connected n-dimensional manifold with $1/4 < K_m \leq 1$, then M is homeomorphic to the n-sphere S^n. In 1977, Grove and Shiohama [GS] proved the generalized sphere theorem which states that a complete n-manifold M with $K_M \geq 1$ and $d_M > \pi/2$ is homeomorphic to S^n.

An elegant theorem due to Myers [M] states that if the Ricci curvature of a complete n-manifold M satisfies that $\mathrm{Ric}_M \geq n-1$, then $d(M) \leq \pi$ and hence M is compact and its fundamental group $\pi_1(M)$ is finite.

In [C], S. Y. Cheng proved the Maximal Diameter Sphere Theorem which states that if $\mathrm{Ric}_M \geq n-1$ and $d_M = \pi$, then M is isometric to the standard sphere S^n. Naturally one will ask if there is a $d_n < \pi$ which depends only on n such that if $\mathrm{Ric}_M \geq n-1$ and $d(M) > d_n$, M is homeomorphic to S^n. Since we can find metrics on $M = S^j \times S^j$ so that $\mathrm{Ric}_M = 2j-1$ and the diameter approaches π as j goes to ∞, for the Ricci curvature case, the dependence on n at least seems inevitable.

This problem is still open. However with some more restrictions on M, Shiohama [S] showed the following volume pinching sphere theorem:

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Theorem. *Let n be a positive integer and let $\kappa > 0$ be a constant. Then there exists an $\varepsilon(n,\kappa) > 0$ such that if M is an n-dimensional complete manifold with $\mathrm{Ric}_M \geq n-1$, $K_M \geq -\kappa^2$ and $v(M) \geq \alpha(n, \pi - \varepsilon(n,\kappa))$, then M is homeomorphic to S^n, where $\alpha(n,r)$ is the volume of the r-ball on $S^n(1)$.*

In this note we generalize this result to the following diameter pinching sphere theorem:

Main Theorem. *Let n be a positive integer and let $\kappa > 0$, $r \in (\pi/2, \pi)$. Then there is δ depending only on n, κ and r such that if M is an n-dimensional complete manifold with $\mathrm{Ric}_M \geq n-1$, $K_M \geq -\kappa^2$, $v(M) \geq \alpha(n,r)$ and $d(M) > \pi - \delta$, M is homeomorphic to S^n.*

The author does not know if the assumption for the sectional curvature and the volume is essentially needed. Under this assumption the radius of contractible metric balls can be bounded away from 0. In our proof we show that if the diameter of M is close to π, the contractibility radius of two particular points which realize the diameter of M will be also close to π. Then we will be able to cover M with two contractible metric balls and appeal to the generalized Schoenflies theorem to complete the proof, where the Generalized Schoenflies Theorem [B] states that if M is covered by two open disks, then M is homeomorphic to S^n.

2. Estimate of contractibility radius

This section is essentially based on the paper [S] of Shiohama. Let M be an n-dimensional manifold. For a fixed point $x \in M$ consider the distance function $d_x : M \to R, d_x(y) = d(x,y)$. A point $y \in M$ is called a critical point of d_x if for any nonzero tangent vector $u \in TM_y$, there is a minimizing geodesic from y to x whose tangent vector at y makes an angle with u not greater than $\pi/2$. Hence a critical point of y of d_x belongs to the cut locus C_x of x.

The contractible radius $c(x)$ at $x \in M$ is defined as

$$c(x) = \mathrm{supp}\{r : \overline{B}_r(x) \text{ is contractible to } x\}.$$

The following two lemmas can be found in [S]:

Lemma 2.1. *For any $x \in M$, if $\overline{B}_r(x)$ contains no critical point of d_x except at the origin x of the ball, then $\overline{B}_r(x)$ is contractible to x. In other words, $c(x)$ is not less than the positive minimum critical value $:= c_1(x)$ of d_x.*

Lemma 2.2. *Let ε be in $(0,\pi)$. Assume that $\mathrm{Ric}_M \geq n-1$ and $v(M) \geq \alpha(n, \pi - \varepsilon)$. For every point $x \in M$ and a number $\theta \in (0,\pi)$ and for every unit tangent vector $u \in SM_x$ let $\Gamma(u,\theta) = \{w \in TM_x : \sphericalangle(u,w) < \theta\}$. Then there exists a positive smooth function $r \to \theta(r,n,\varepsilon), 0 < r < \pi - \varepsilon$ such that if every $w \in \Gamma(u,\theta) \cap C_x$ has norm $\|w\| \leq r$, then $\theta \leq \theta(r,n,\varepsilon)$. $\theta(r,n,\varepsilon)$ is* *obtained as the solution of*

$$c_{n-2}\int_0^{\theta(r,n,\varepsilon)} \sin^{n-2} t\,dt \int_r^{\pi} \sin^{n-1} t\,dt = \alpha(n,\varepsilon),$$

where c_m *is the volume of* $S^m(1)$.

Remark. For $\delta \in (\varepsilon, \pi/2), \theta(\pi-2\delta, n, \varepsilon) < \pi/2$.

We are now in a position to estimate $c_1(x)$.

Theorem 2.3. *Let* n *be a positive integer and let* $\kappa \geq 0$ *and* $\varepsilon \in (0,\pi/2)$ *be given. Then there exists for a fixed number* $\delta \in (\varepsilon, \pi/2)$ *a constant* $c_\delta(n,\kappa,\varepsilon) > 0$ *such that if* M *is a complete n-dimensional manifold with*

$$\mathrm{Ric}_M \geq n-1, \qquad K_M \geq -\kappa^2, \qquad v(M) \geq \alpha(n, \pi-\varepsilon),$$

Then $c_1(x) \geq c_\delta(n,\kappa,\varepsilon)$ *for every point* $x \in M$. *The constant is given by*

$$c_\delta(n,\kappa,\varepsilon) = \min\{\pi-2\delta, \kappa^{-1}\tanh^{-1}[\tanh(\pi-2\delta)\kappa\cos\theta(\pi-2\delta,n,\varepsilon)]\}.$$

Proof. Let $r_1 = \pi - 2\delta$ and let $x \in M$ be a fixed point and y a critical point of d_x with the positive minimum critical value $r_0 = c_1(x)$. Let $u \in SM_x$ be the unit vector tangent to a minimizing geodesic $\gamma_u\colon [0, r_0] \to M$ with $\gamma_u(0) = x$, $\gamma_u(r_0) = y$.

By the above lemma and the continuity of the map $w \in SM_x \to$ the distance from x to the cut point of x along the geodesic $t \to \exp_x tw$, there is a $w \in SM_x$ with the properties $\sphericalangle(u,w) \leq \theta(r_1, n, \varepsilon)$ and γ_w has the cut point to x along it at $\gamma_w(t_1)$ with $t_1 \geq r_1$. If $r_0 \geq r_1$, then we are done. Hence we can assume that $r_0 < r_1$. The Toponogov Comparison Theorem implies that if $\alpha = \sphericalangle(u,w)$ and if $r_2 = d(y,z)$ where $z = \gamma_w(t_1)$, then

$$\cosh r_2\kappa \leq \cosh t_1\kappa \cosh r_0\kappa - \sinh t_1\kappa \sinh r_0\kappa \cos\alpha.$$

Since y is a critical point of d_x, there is, for a minimizing geodesic from y to z, a minimizing geodesic from y to x (possibly different from γ_u) whose angle at y is not greater than $\pi/2$. Thus again by the Toponogov Comparison Theorem, one has

$$\cosh t_1\kappa \leq \cosh r_0\kappa \cosh r_2\kappa.$$

Eliminate r_2 from the above inequalities to obtain

$$\cosh t_1\kappa \tanh r_0\kappa \geq \cos\alpha.$$

Insert $\alpha \leq \theta(r_1, n, \varepsilon) < \pi/2$ and $t_1 \geq r_1 = \pi - 2\delta$ to complete the proof.

Remark. This theorem is basically due to Shiohama. In [S], he proved this for $\varepsilon \in (0, \pi/3)$.

3. The proof of Main Theorem

Before we start to prove the main theorem let's recall the Bishop-Gromov Volume Comparison Theorem [G].

Let M be a complete manifold of dimension n with $\mathrm{Ric}_M \geq -(n-1)\kappa^2$, where κ is a real or a pure imaginary number. Let $M(-\kappa^2)$ be the complete simply connected n-dimensional space form of constant sectional curvature $-\kappa^2$. For a point $x \in M$ and for an $r > 0$ let $B_r(x)$ be the metric r-ball centered at x. A metric r-ball in $M(-\kappa^2)$ is denoted by $\widetilde{B}_r$. With these notations the Bishop-Gromov Volume Comparison Theorem is stated as

Lemma 3.1. *For any fixed $x \in M$ and $0 \leq r \leq R$,*

$$\frac{v(B_r(x))}{v(B_R(x))} \geq \frac{v(\widetilde{B}_r)}{v(\widetilde{B}_R)}.$$

Let $\{M_k\}$ be a sequence of complete manifolds with $\mathrm{Ric}_{M_k} \geq n-1$, $K_{M_k} \geq -\kappa^2$ and $v(M_k) \geq \alpha(n, r)$ where κ, r are as in the Main Theorem and assume that $d_k = d(M_k) \to \pi$ as $k \to \infty$.

Let p_k and q_k be in M_k with $d(p_k, q_k) = d_k$. Now we are going to investigate the contractibility radius of p_k and q_k. Choose $y_k \in M_k$ to be a critical point of d_{p_k} with the positive minimum critical value $r_k = c_1(p_k)$ and let $t_k = d(y_k, q_k)$. By Theorem 2.3, $r_k \geq r_0$ for some positive number r_0. Without loss of generality, we can assume that $\lim r_k = \alpha \geq r_0$ and $\lim t_k = \beta$. Since $r_k + t_k \geq d_k$, $\alpha + \beta \geq \pi$.

Claim 1. $\beta = \pi - \alpha$.

Proof. Supposing this is not true, one can find a positive number $\varepsilon < 1/4 \min\{r_0, \alpha + \beta - \pi\}$ and N_0 such that if $k \geq N_0$, then $t_k \geq \pi - \alpha + 3\varepsilon$ and $\alpha - \varepsilon < r_k < \alpha + \varepsilon$. Hence $t_k > d_k - r_k + 2\varepsilon$. Thus the balls $B_{p_k}(r_k - \varepsilon)$, $B_{q_k}(d_k - r_k + \varepsilon)$ and $B_{y_k}(\varepsilon)$ are pairwise disjoint in M_k. This gives

$$v(M_k) \geq v(B_{p_k}(r_k - \varepsilon)) + v(B_{q_k}(d_k - r_k + \varepsilon)) + v(B_{y_k}(\varepsilon)).$$

Dividing by $v(M_k)$ on both sides and using the Bishop-Gromov Volume Comparison Theorem, one has

$$1 \geq [\alpha(n, r_k - \varepsilon) + \alpha(n, d_k - r_k + \varepsilon) + \alpha(n, \varepsilon)]/\alpha(n, \pi).$$

Now letting $k \to \infty$,

$$1 \geq 1 + \alpha(n, \varepsilon)/\alpha(n, \pi).$$

This is impossible, hence $\beta = \pi - \alpha$.

Claim 2. $\alpha = \pi$.

Proof. Since y_k is a critical point of d_{p_k} there exists, for a minimizing geodesic from y_k to q_k a minimizing geodesic from y_k to p_k whose angle at y_k is not greater than $\pi/2$. Thus the Toponogov Comparison Theorem applies for this triangle to give

$$\cosh d_k\kappa \leq \cosh r_k\kappa \cosh t_k\kappa .$$

Letting $k \to \infty$ and using Claim 1,

$$\cosh \pi\kappa \leq \cosh \alpha\kappa \cosh(\pi - \alpha)\kappa .$$

This gives that $\alpha = \pi$.

The Proof of Main Theorem. Suppose that Main Theorem is false. Then there exists a sequence of manifolds M_k which are not homeomorphic to S^n such that $\mathrm{Ric}_{M_k} \geq n-1$, $K_{M_k} \geq -\kappa^2$, $v(M_k) \geq \alpha(n, r)$ and $d_k = d(M_k) \to \pi$. Let p_k and q_k be in M_k with $d(p_k, q_k) = d_k$. By the above argument and Lemma 2.1, the contractibility radii $c(p_k)$ and $c(q_k)$ are greater than $2\pi/3$ for large k.

The minimal radius R_k of closed balls around p_k and q_k by which M_k is covered satisfies $d_k/2 \leq R_k \leq d_k$ and $R_k = \max\{d(p_k, x) \colon x \in M_k, d(p_k, x) = d(x, q_k)\}$. If $x_k \in M_k$ is a point with $d(p_k, x_k) = d(x_k, q_k) = R_k$, then

$$v(M_k) \geq v(B_{p_k}(d_k/2)) + v(B_{q_k}(d_k/2)) + v(B_{x_k}(R_k - d_k/2)) .$$

Dividing by $v(M_k)$ and again using the Bishop-Gromov Volume Comparison Theorem ,

$$1 \geq [2\alpha(n, d_k/2) + \alpha(n, R_k - d_k/2)]/\alpha(n, \pi) .$$

Since $d_k \to \pi$, we conclude that $R_k \to \pi/2$. Hence for large k, $R_k < 2\pi/3$. Therefore for large k, M_k can be covered by two contractible metric balls $B_{p_k}(2\pi/3)$ and $B_{q_k}(2\pi/3)$. The Generalized Schoenflies Theorem implies that M_k is homeomorphic to S^n. This desired contradiction completes the proof.

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