BONFERRONI-TYPE INEQUALITIES
VIA INTERPOLATING POLYNOMIALS

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Abstract. Inequalities of Bonferroni-type are proved by means of the corresponding polynomial inequalities which are obtained from the Hermite interpolation.

1. Introduction

Let $A_1, A_2, \ldots, A_n$ be events of a given probability space, and set $S_{0,n} = 1$,

$$S_{k,n} = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}), \quad k \geq 1,$$

where the summation is over $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The numbers $S_{k,n}$ are the binomial moments of the number $m_n$ of occurrences amount the $A_j$.

Linear inequalities of the form

$$\sum c_k S_{k,n} \leq P(m_n = 0) \leq \sum d_k S_{k,n} \leq \sum d_k \binom{n}{k} p^k$$

(1)

are equivalent to the polynomial inequalities

$$\sum c_k \binom{n}{k} p^k \leq (1 - p)^n \leq \sum d_k \binom{n}{k} p^k$$

by a result of Galambos and Mucci [3]. The classical Bonferroni inequalities corresponding to $c_k = (-1)^k$, $1 \leq k \leq 2r - 1$, and $c_k = 0$ for $k \geq 2r$, and $d_k = (-1)^k$, $1 \leq k \leq 2r$, and $d_k = 0$ for $k \geq 2r + 1$. The corresponding polynomial inequality (2) is just a finite Taylor expansion of $(1 - p)^n$, or from another point of view, this is the Hermite interpolating polynomial at one point $p = 0$. This fact raises the question whether other interpolating polynomials provide better inequalities in (2) and thus in (1). This paper deals with such interpolating polynomial methods. In particular, we will show that the Sobel-Uppuluri-Galambos inequalities

$$\sum_{k=1}^{2r} (-1)^{k+1} S_{k,n} + \frac{2t+1}{n} S_{2t+1,n} \leq P(m_n = 0) \leq \sum_{k=1}^{2t-1} (-1)^{k+1} S_{k,n} - \frac{2t}{n} S_{2t,n}$$

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2. Hermite Interpolating Polynomials

For a given function \( f \), we consider the two-point Hermite interpolating polynomial \( H_{r,s}(f; p) \) of degree \( \leq r + s - 1 \), which is uniquely defined by the following conditions:

\[
H_{r,s}^{(q)}(f; 0) = f^{(q)}(0), \quad 0 \leq q \leq r - 1,
\]
\[
H_{r,s}^{(q)}(f; 1) = f^{(q)}(1), \quad 0 \leq q \leq s - 1.
\]

The polynomial \( H_{r,s}(f; p) \) can conveniently be written in the form

\[
H_{r,s}(f; p) = \sum_{k=0}^{r-1} T_k(p)f^{(k)}(0) + \sum_{k=0}^{s-1} R_k(p)f^{(k)}(1),
\]

where \( T_k(p) \) is uniquely determined by

\[
T_k^{(j)}(0) = \delta_{jk}, \quad 0 \leq j \leq r - 1; \quad T_k^{(j)}(1) = 0, \quad 0 \leq j \leq s - 1.
\]

From (5), it can be easily verified that

\[
T_k(p) = (1 - p)^{s-r} \sum_{k=0}^{r-k-1} \binom{s + i - 1}{i} p^{i+k}/k!.
\]

Throughout this paper, we use notations \( (q)_0 = 1 \) and \( (q)_p = 0 \), \( q < p \), or \( p < 0 \). Now let \( f_n(p) = (1 - p)^n \); we can write \( H_{r,s}(f_n; p) \) into a more compact form.

**Lemma 1.** \( H_{r,s+1}(f_n; p) = (1 - p)H_{r,s}(f_{n-1}; p) \).

**Proof.** By (4) and (6)

\[
(1 - p)^{-s-1}H_{r,s+1}(f_n; p) = \sum_{k=0}^{r-1} (-1)^k \binom{n}{k} \sum_{i=0}^{r-k-1} \binom{s + i - 1}{i} p^{i+k}
\]

\[
= \sum_{k=0}^{r-1} (-1)^k \binom{n - 1}{k} \sum_{i=0}^{r-k-1} \binom{s + i - 1}{i} p^{k+i}
\]

\[
+ \sum_{k=0}^{r-1} (-1)^k \binom{n - 1}{k} \sum_{i=0}^{r-k-1} \binom{s + i - 1}{i} p^{k+i}
\]

\[
+ \sum_{k=0}^{r-1} (-1)^k \binom{n - 1}{k - 1} \sum_{i=0}^{r-k-1} \binom{s + 1}{i} p^{i+k}
\]

\[
= I_1 + I_2 + I_3,
\]
where \( \binom{m}{j} = \binom{m-1}{j} \binom{n-1}{j-1} \) is used. Since it can be easily checked that \( I_2 = -I_3 \),
we get

\[
(1 - p)^{-s-1} H_{r,s+1}(f_n; p) = \sum_{k=0}^{r-1} (-1)^k \binom{n-1}{k} \sum_{i=0}^{r-k-1} \binom{s + i - 1}{i} p^{k+i}
\]

\[
= (1 - p)^{-s} H_{r,s}(f_{n-1}; p).
\]

Using Lemma 1 repeatedly, we get

\[
H_{r,s}(f_n; p) = (1 - p)^s H_{r,0}(f_{n-s}; p),
\]

where by (4) and (5),

\[
H_{r,0}(f_{n-s}; p) = \sum_{k=0}^{r-1} (-1)^k \binom{n-s}{k} p^k.
\]

Furthermore, we get from (7) and (8) that

\[
H_{r,s+1}(f_n; p) - H_{r,s}(f_n; p) = (1 - p)^s (-1)^r \binom{n-s-1}{r-1} p^r;
\]

therefore,

\[
H_{2m,s+1}(f_n; p) \leq H_{2m,s}(f_n; p), \quad H_{2m+1,s+1}(f_n; p) \leq H_{2m+1,s}(f_n; p).
\]

From the remainder formula of Hermite interpolation (cf. [1]), we get

\[
f_n(p) - H_{r,s}(f_n; p) = \frac{p^r (p - 1)^s}{(r + s)!} f_n^{(r+s)}(\xi)
\]

\[
= (-1)^r \binom{n}{r+s} p^r (1 - p)^s (1 - \xi)^{n-r-s}, \quad 0 < \xi < 1.
\]

Therefore, we have polynomial inequalities

\[
H_{2m,s}(f_n; p) \leq f_n(p) = (1 - p)^n \leq H_{2m+1,s}(f_n; p).
\]

Consequently, we have corresponding Bonferroni-type inequalities, if we know explicitly the coefficients of monomials in the expression of \( H_{r,s}(f_n; p) \). For
a mixed $m$, it follows from (9) that we get sharper inequalities (10) and thus sharper Bonferroni-type inequalities as $s$ increases.

3. Bonferroni-type inequalities

We use Newton's binomial equality to write $H_{r,s}(f_n;p)$ as sum of monomials. If $r \geq s$, then from (7) and (8),

$$H_{r,s}(f_n;p) = \sum_{j=0}^{s} \binom{s}{j} (-1)^j p^j \sum_{k=0}^{r-1} (-1)^k \binom{n-s}{k} p^k$$

$$= \sum_{j=0}^{s} \binom{s}{j} \sum_{k=j}^{r-1} (-1)^k \binom{n-s}{k-j} p^k$$

$$= \sum_{j=0}^{s} \binom{s}{j} \sum_{k=j}^{r-1} (-1)^k \binom{n-s}{k-j} p^k + \sum_{j=0}^{s} \binom{s}{j} \sum_{k=r}^{s} (-1)^k \binom{n-s}{k-j} p^k$$

$$:= J_1 + J_2.$$  

Interchanging summations, and then applying Euler's formula

$$\sum_{i=0}^{m} \binom{m}{i} \binom{n}{k-i} = \binom{n+m}{k},$$

we obtain

$$J_1 = \sum_{k=0}^{r-1} (-1)^k p^k \binom{n}{k}.$$ 

Furthermore, by interchanging summations, we have

$$J_2 = \sum_{j=0}^{s} \binom{s}{j} \sum_{k=0}^{j-1} (-1)^{k+r} \binom{n-s}{k+r-j} p^{k+r}$$

$$= \sum_{k=0}^{s-1} (-1)^{k+r} p^{k+r} \sum_{j=1}^{s-k} \binom{s}{j+k} \binom{n-s}{r-j}.$$ 

Therefore, if $r \geq s$, then

$$H_{r,s}(f_n;p) = \sum_{k=0}^{r-1} (-1)^k \binom{n}{k} p^k + \sum_{k=0}^{s-1} (-1)^{k+r} p^{k+r} \sum_{j=1}^{s-k} \binom{s}{j+k} \binom{n-s}{r-j}.$$ 

Now, by (9) and (10), Theorem 1 follows from the correspondence of (1) and (2).
Theorem 1. For positive integers \( m \), \( s \) and \( n \geq 1 \), \( 2m + s - 1 < n \),

\[
\sum_{k=0}^{2m-1} (-1)^k S_{k,n} + \sum_{k=0}^{s-1} (-1)^k \left[ \sum_{j=1}^{s-k} \binom{s}{j+k} \left( \frac{n-s}{2m-j} \right) \binom{n}{k+2m} \right] S_{k+2m,n}
\]

\[
\leq P(m_n = 0) \leq \sum_{k=0}^{2m} (-1)^k S_{k,n}
\]

\[
- \sum_{k=0}^{s-1} (-1)^k \left[ \sum_{j=1}^{s-k} \binom{s}{j+k} \left( \frac{n-s}{2m+1-j} \right) \binom{n}{k+2m+1} \right] S_{k+2m+1,n}.
\]

Furthermore, the above inequalities become sharper as \( s \) increases.

For \( s = 0 \), Theorem 1 reduces to the Bonferroni inequalities, and for \( s = 1 \), it reduces to the Sobel-Uppuluri-Galambos inequalities (3). We list the next case \( s = 2 \) as the following corollary.

Corollary 1. For integers \( n \geq 1 \), \( 1 \leq 2m + 1 < n \),

\[
\sum_{k=0}^{2m-1} (-1)^k S_{k,n} + \frac{2m(2n-2m-1)}{n(n-1)} S_{2m,n} - \frac{4m(2m+1)}{n(n-1)} S_{2m+1,n}
\]

\[
\leq P(m_n = 0) \leq \sum_{k=0}^{2m} (-1)^k S_{k,n}
\]

\[
- \frac{(2m+1)(2n-2m-2)}{n(n-1)} S_{2m+1,n} + \frac{4(m+1)(2m+1)}{n(n-1)} S_{2m+2,n}.
\]

We now consider the case of \( r \leq s \). By (4) and (6), we have

\[
H_{r,s}(f_n;p) = \sum_{k=0}^{r-1} \binom{n-s}{k} \sum_{j=0}^{s+k} \binom{s}{j+k} (-1)^j p^j
\]

\[
= \sum_{k=0}^{r-1} \binom{n-s}{k} \sum_{j=0}^{r-1} \binom{s}{j+k} (-1)^j p^j
\]

\[
+ \sum_{k=0}^{r-1} \binom{n-s}{k} \sum_{j=r}^{s+k} \binom{s}{j+k} (-1)^j p^j
\]

\[
:= J_1 + J_2,
\]

where interchanging summations and applying Euler's formula, we obtain

\[
J_1 = \sum_{j=0}^{r-1} (-1)^j \binom{n}{j} p^j,
\]

and

\[
J_2 = \sum_{j=r}^{s-1} (-1)^j p^j \sum_{k=0}^{r-1} \binom{n-s}{k} \binom{s}{j-k} + \sum_{k=0}^{r-1} \binom{n-s}{k} \sum_{j=s}^{s+k} (-1)^j \binom{s}{j-k} p^j.
\]
where the second term in the right-hand side
\[
= \sum_{j=s}^{r+s-1} (-1)^j p^j \sum_{k=j-s}^{r-1} \binom{n-s}{k} \binom{s}{j-k}
\]

\[
= \sum_{j=s}^{r+s-1} (-1)^j p^j \sum_{k=0}^{r-1} \binom{n-s}{k} \binom{s}{j-k},
\]
since if \( k < j-s \), then \( \binom{s}{j-k} = \binom{s}{s-j+k} = 0 \). So
\[
J_2 = \sum_{j=r}^{s+r-1} (-1)^j p^j \left[ \sum_{k=0}^{r-1} \binom{n-s}{k} \binom{s}{j-k} \right].
\]

Therefore, if \( r \leq s \), then
\[
H_{r,s}(f_n;p) = \sum_{j=0}^{r-1} (-1)^j \binom{n}{j} p^j + \sum_{j=r}^{s+r-1} (-1)^j p^j \left[ \sum_{k=0}^{r-1} \binom{n-s}{k} \binom{s}{j-k} \right].
\]

By (9) and (10), and the correspondence between (1) and (2), we then have

**Theorem 2.** For integers \( m, s \) and \( n \), \( 0 \leq 2m < s \), \( 2m + s - 1 < n \),
\[
\sum_{j=0}^{2m+1} (-1)^j S_{j,n} + \sum_{j=2m+2}^{s+2m+1} (-1)^j \left[ \sum_{k=0}^{2m+1} \binom{s}{j-k} \binom{n-s}{k} / \binom{n}{j} \right] S_{j,n}
\]
\[
\leq P(m_n = 0)
\]
\[
\leq \sum_{j=0}^{2m} (-1)^j S_{j,n} + \sum_{j=2m+1}^{s+2m} (-1)^j \left[ \sum_{j=0}^{2m} \binom{s}{j-k} \binom{n-s}{k} / \binom{n}{j} \right] S_{j,n}.
\]

We list the case corresponding to \( m = 0 \) as the following corollary.

**Corollary 2.** For \( s \geq 2 \),
\[
\sum_{j=0}^{s} (-1)^j \binom{s}{j} + \binom{n-s}{j-1} S_{j,n} + (-1)^{s+1} \frac{n-s}{n-s+1} S_{j,n}
\]
\[
\leq P(m_n = 0) \leq \sum_{j=0}^{s} (-1)^j \frac{s_j}{j!} S_{j,n}.
\]

For \( s = 2 \), Corollary 2 reduces to
\[
1 - S_{1,n} + 2 \frac{2n-3}{n(n-1)} S_{2,n} - \frac{6}{n(n-1)} S_{3,n} \leq P(m_n = 0)
\]
\[
\leq 1 - \frac{2}{n} S_{1,n} + \frac{2}{n(n-1)} S_{2,n},
\]
where the right-hand side is well known [2, Theorem 1.4.3, \( k = n-1 \)].
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