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BONFERRONI-TYPE INEQUALITIES VIA INTERPOLATING POLYNOMIALS

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ABSTRACT. Inequalitites of Bonferroni-type are proved by means of the corresponding polynomial inequalities which are obtained from the Hermite interpolation.

1. INTRODUCTION

Let $A_1, A_2, \ldots, A_n$ be events of a given probability space, and set $S_{0,n} = 1$,

$$S_{k,n} = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}), \qquad k \geqq 1,$$

where the summation is over $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$. The numbers $S_{k,n}$ are the binomial moments of the number m_n of occurrences amount the A_j. Linear inequalities of the form

$$\sum c_k S_{k,n} \leqq P(m_n = 0) \leqq \sum d_k S_{k,n} \tag{1}$$

are equivalent to the polynomial inequalities

$$\sum c_k \binom{n}{k} p^k \leqq (1-p)^n \leqq \sum d_k \binom{n}{k} p^k \tag{2}$$

by a result of Galambos and Mucci [3]. The classical Bonferroni inequalities corresponding to $c_k = (-1)^k$, $1 \leqq k \leqq 2r-1$, and $c_k = 0$ for $k \geqq 2r$, and $d_k = (-1)^k$, $1 \leqq k \leqq 2r$, and $d_k = 0$ for $k \geqq 2r+1$. The corresponding polynomial inequality (2) is just a finite Taylor expansion of $(1-p)^n$, or from another point of view, this is the Hermite interpolating polynomial at one point $p = 0$. This fact raises the question whether other interpolating polynomials provide better inequalities in (2) and thus in (1). This paper deals with such interpolating polynomial methods. In particular, we will show that the Sobel-Uppuluri-Galambos inequalities

$$\sum_{k=1}^{2t} (-1)^{k+1} S_{k,n} + \frac{2t+1}{n} S_{2t+1,n} \leqq P(m_n = 0) \leqq \sum_{k=1}^{2t-1} (-1)^{k+1} S_{k,n} - \frac{2t}{n} S_{2t,n} \tag{3}$$

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[2, Theorem 1.41] are given by a special two-point Hermite interpolating polynomial.

2. Hermite interpolating polynomials

For a given function f, we consider the two-point Hermite interpolating polynomial $H_{r,s}(f;p)$ of degree $\leqq r+s-1$, which is uniquely defined by the following conditions:

$$H_{r,s}^{(q)}(f;0) = f^{(q)}(0), \qquad 0 \leqq q \leqq r-1,$$
$$H_{r,s}^{(q)}(f;1) = f^{(q)}(1), \qquad 0 \leqq q \leqq s-1.$$

The polynomial $H_{r,s}(f;p)$ can conveniently be written in the form

$$H_{r,s}(f;p) = \sum_{k=0}^{r-1} T_k(p) f^{(k)}(0) + \sum_{k=0}^{s-1} R_k(p) f^{(k)}(1), \tag{4}$$

where $T_k(p)$ is uniquely determined by

$$T_k^{(j)}(0) = \delta_{jk}, \qquad 0 \leqq j \leqq r-1; \qquad T_k^{(j)}(1) = 0, \qquad 0 \leqq j \leqq s-1. \tag{5}$$

From (5), it can be easily verified that

$$T_k(p) = (1-p)^s \sum_{k=0}^{r-k-1} \binom{s+i-1}{i} p^{i+k}/k!. \tag{6}$$

Throughout this paper, we use notations $\binom{q}{0} = 1$ and $\binom{q}{p} = 0$, $q < p$, or $p < 0$. Now let $f_n(p) = (1-p)^n$; we can write $H_{r,s}(f_n;p)$ into a more compact form.

Lemma 1. $H_{r,s+1}(f_n;p) = (1-p)H_{r,s}(f_{n-1};p)$.

Proof. By (4) and (6)

$$\begin{aligned}
(1-p)^{-s-1} H_{r,s+1}(f_n;p) &= \sum_{k=0}^{r-1} (-1)^k \binom{n}{k} \sum_{i=0}^{r-k-1} \binom{s+i}{i} p^{i+k} \\
&= \sum_{k=0}^{r-1} (-1)^k \binom{n-1}{k} \sum_{i=0}^{r-k-1} \binom{s+i-1}{i} p^{k+i} \\
&\quad + \sum_{k=0}^{r-1} (-1)^k \binom{n-1}{k} \sum_{i=0}^{r-k-1} \binom{s+i-1}{i-1} p^{k+i} \\
&\quad + \sum_{k=0}^{r-1} (-1)^k \binom{n-1}{k-1} \sum_{i=0}^{r-k-1} \binom{s+1}{i} p^{i+k} \\
&:= I_1 + I_2 + I_3,
\end{aligned}$$

where $\binom{m}{j} = \binom{m-1}{j}\binom{m-1}{j-1}$ is used. Since it can be easily checked that $I_2 = -I_3$, we get

$$(1-p)^{-s-1} H_{r,s+1}(f_n;p) = \sum_{k=0}^{r-1} (-1)^k \binom{n-1}{k} \sum_{i=0}^{r-k-1} \binom{s+i-1}{i} p^{k+i}$$
$$= (1-p)^{-s} H_{r,s}(f_{n-1};p).$$

Using Lemma 1 repeatedly, we get

$$H_{r,s}(f_n;p) = (1-p)^s H_{r,0}(f_{n-s};p), \tag{7}$$

where by (4) and (5),

$$H_{r,0}(f_{n-s};p) = \sum_{k=0}^{r-1} (-1)^k \binom{n-s}{k} p^k. \tag{8}$$

Furthermore, we get from (7) and (8) that

$$H_{r,s+1}(f_n;p) - H_{r,s}(f_n;p) = (1-p)^s (-1)^r \binom{n-s-1}{r-1} p^r;$$

therefore,

$$H_{2m,s+1}(f_n;p) \geqq H_{2m,s}(f_n;p), \qquad H_{2m+1,s+1}(f_n;p) \leqq H_{2m+1,s}(f_n;p). \tag{9}$$

From the remainder formula of Hermite interpolation (cf. [1]), we get

$$f_n(p) - H_{r,s}(f_n;p) = \frac{p^r (p-1)^s}{(r+s)!} f_n^{(r+s)}(\zeta)$$
$$= (-1)^r \binom{n}{r+s} p^r (1-p)^s (1-\zeta)^{n-r-s}, \qquad 0 < \zeta < 1.$$

Therefore, we have polynomial inequalities

$$H_{2m,s}(f_n;p) \leqq f_n(p) = (1-p)^n \leqq H_{2m+1,s}(f_n;p). \tag{10}$$

Consequently, we have corresponding Bonferroni-type inequalities, if we know explicitly the coefficients of monomials in the expression of $H_{r,s}(f_n;p)$. For

a mixed m, it follows from (9) that we get sharper inequalities (10) and thus sharper Bonferroni-type inequalities as s increases.

3. Bonferroni-type inequalities

We use Newton's binomial equality to write $H_{r,s}(f_n;p)$ as sum of monomials. If $r \geqq s$, then from (7) and (8),

$$\begin{aligned} H_{r,s}(f_n;p) &= \sum_{j=0}^{s}\binom{s}{j}(-1)^j p^j \sum_{k=0}^{r-1}(-1)^k\binom{n-s}{k}p^k \\ &= \sum_{j=0}^{s}\binom{s}{j}\sum_{k=j}^{r-1+j}(-1)^k\binom{n-s}{k-j}p^k \\ &= \sum_{j=0}^{s}\binom{s}{j}\sum_{k=j}^{r-1}(-1)^k\binom{n-s}{k-j}p^k + \sum_{j=0}^{s}\binom{s}{j}\sum_{k=r}^{r-1+j}(-1)^k\binom{n-s}{k-j}p^k \\ &:= J_1 + J_2. \end{aligned}$$

Interchanging summations, and then applying Euler's formula

$$\sum_{i=0}^{m}\binom{m}{i}\binom{n}{k-i} = \binom{n+m}{k},$$

we obtain

$$J_1 = \sum_{k=0}^{r-1}(-1)^k p^k \binom{n}{k}.$$

Furthermore, by interchanging summations, we have

$$\begin{aligned} J_2 &= \sum_{j=0}^{s}\binom{s}{j}\sum_{k=0}^{j-1}(-1)^{k+r}\binom{n-s}{k+r-j}p^{k+r} \\ &= \sum_{k=0}^{s-1}(-1)^{k+r}p^{k+r}\left[\sum_{j=1}^{s-k}\binom{s}{j+k}\binom{n-s}{r-j}\right]. \end{aligned}$$

Therefore, if $r \geqq s$, then

$$H_{r,s}(f_n;p) = \sum_{k=0}^{r-1}(-1)^k\binom{n}{k}p^k + \sum_{k=0}^{s-1}(-1)^{k+r}p^{k+r}\left[\sum_{j=1}^{s-k}\binom{s}{j+k}\binom{n-s}{r-j}\right].$$

Now, by (9) and (10), Theorem 1 follows from the correspondence of (1) and (2).

Theorem 1. *For positive integers* m, s *and* $n \geqq 1$, $2m+s-1<n$,

$$\sum_{k=0}^{2m-1}(-1)^k S_{k,n}+\sum_{k=0}^{s-1}(-1)^k\left[\sum_{j=1}^{s-k}\binom{s}{j+k}\binom{n-s}{2m-j}\Big/\binom{n}{k+2m}\right]S_{k+2m,n}$$
$$\leqq P(m_n=0)\leqq\sum_{k=0}^{2m}(-1)^k S_{k,n}$$
$$-\sum_{k=0}^{s-1}(-1)^k\left[\sum_{j=1}^{s-k}\binom{s}{j+k}\binom{n-s}{2m+1-j}\Big/\binom{n}{k+2m+1}\right]S_{k+2m+1,n}.$$

Furthermore, the above inequalities become sharper as s *increases.*

For $s=0$, Theorem 1 reduces to the Bonferroni inequalities, and for $s=1$, it reduces to the Sobel-Uppuluri-Galambos inequalities (3). We list the next case $s=2$ as the following corollary.

Corollary 1. *For integers* $n\geqq 1$, $1\leqq 2m+1<n$,

$$\sum_{k=0}^{2m-1}(-1)^k S_{k,n}+\frac{2m(2n-2m-1)}{n(n-1)}S_{2m,n}-\frac{4m(2m+1)}{n(n-1)}S_{2m+1,n}$$
$$\leqq P(m_n=0)\leqq\sum^{2m}(-1)^k_{k=0}S_{k,n}$$
$$-\frac{(2m+1)(2n-2m-2)}{n(n-1)}S_{2m+1,n}+\frac{4(m+1)(2m+1)}{n(n-1)}S_{2m+2,n}.$$

We now consider the case of $r\leqq s$. By (4) and (6), we have

$$\begin{aligned}H_{r,s}(f_n;p)&=\sum_{k=0}^{r-1}\binom{n-s}{k}\sum_{j=k}^{s+k}\binom{s}{j-k}(-1)^j p^j\\&=\sum_{k=0}^{r-1}\binom{n-s}{k}\sum_{j=k}^{r-1}\binom{s}{j-k}(-1)^j p^j\\&\quad+\sum_{k=0}^{r-1}\binom{n-s}{k}\sum_{j=r}^{s+k}\binom{s}{j-k}(-1)^j p^j\\&:=J_1+J_2,\end{aligned}$$

where interchanging summations and applying Euler's formula, we obtain

$$J_1=\sum_{j=0}^{r-1}(-1)^j\binom{n}{j}p^j,$$

and

$$J_2=\sum_{j=r}^{s-1}(-1)^j p^j\sum_{k=0}^{r-1}\binom{n-s}{k}\binom{s}{j-k}+\sum_{k=0}^{r-1}\binom{n-s}{k}\sum_{j=s}^{s+k}(-1)^j\binom{s}{j-k}p^j,$$

where the second term in the right-hand side

$$= \sum_{j=s}^{r+s-1} (-1)^j p^j \sum_{k=j-s}^{r-1} \binom{n-s}{k}\binom{s}{j-k}$$
$$= \sum_{j=s}^{r+s-1} (-1)^j p^j \sum_{k=0}^{r-1} \binom{n-s}{k}\binom{s}{j-k},$$

since if $k < j-s$, then $\binom{s}{j-k} = \binom{s}{s-j+k} = 0$. So

$$J_2 = \sum_{j=r}^{s+r-1} (-1)^j p^j \left[\sum_{k=0}^{r-1} \binom{n-s}{k}\binom{s}{j-k}\right].$$

Therefore, if $r \leqq s$, then

$$H_{r,s}(f_n;p) = \sum_{j=0}^{r-1}(-1)^j \binom{n}{j} p^j + \sum_{j=r}^{s+r-1} (-1)^j p^j \left[\sum_{k=0}^{r-1} \binom{n-s}{k}\binom{s}{j-k}\right].$$

By (9) and (10), and the correspondence between (1) and (2), we then have

Theorem 2. *For integers m, s and n, $0 \leqq 2m < s$, $2m+s-1 < n$,*

$$\sum_{j=0}^{2m+1} (-1)^j S_{j,n} + \sum_{j=2m+2}^{s+2m+1} (-1)^j \left[\sum_{k=0}^{2m+1} \binom{s}{j-k}\binom{n-s}{k} \Big/ \binom{n}{j}\right] S_{j,n}$$
$$\leqq P(m_n = 0)$$
$$\leqq \sum_{j=0}^{2m}(-1)^j S_{j,n} + \sum_{j=2m+1}^{s+2m} (-1)^j \left[\sum_{j=0}^{2m} \binom{s}{j-k}\binom{n-s}{k} \Big/ \binom{n}{j}\right] S_{j,n}.$$

We list the case corresponding to $m = 0$ as the following corollary.

Corollary 2. *For $s \geqq 2$,*

$$\sum_{j=0}^{s}(-1)^j \frac{\binom{s}{j} + (n-s)\binom{s}{j-1}}{\binom{n}{j}} S_{j,n} + (-1)^{s+1} \frac{n-s}{\binom{n}{s+1}} S_{j,n}$$
$$\leqq P(m_n = 0) \leqq \sum_{j=0}^{s}(-1)^j \frac{\binom{s}{j}}{\binom{n}{j}} S_{j,n}.$$

For $s = 2$, Corollary 2 reduces to

$$1 - S_{1,n} + 2\frac{2n-3}{n(n-1)} S_{2,n} - \frac{6}{n(n-1)} S_{3,n} \leqq P(m_n = 0)$$
$$\leqq 1 - \frac{2}{n} S_{1,n} + \frac{2}{n(n-1)} S_{2,n},$$

where the right-hand side is well known [2, Theorem 1.4.3, $k = n-1$].

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