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SMOOTH POLYNOMIAL PATHS WITH NONANALYTIC TANGENTS

ROBERT M. MCLEOD AND GARY H. MEISTERS

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ABSTRACT. We prove that there exist C^∞ functions $\varphi\colon \mathbf{R}_t \times \mathbf{R}_x \to \mathbf{R}$ such that although $\varphi(t,x)$ is a polynomial in x for each t in $\mathbf{R}$, $\dot\varphi(0,x) \equiv (\partial\varphi/\partial t)(0,x)$ need not even be analytic in x let alone polynomial. It was shown earlier by one of the authors [Meisters] that this cannot happen if φ satisfies the group-property (even locally) of flows, namely if $\varphi(s,\varphi(t,x)) = \varphi(s+t,x)$.

If a class C^1 function $\varphi\colon \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ satisfies the group property

$$\varphi(s,\varphi(t,x)) = \varphi(s+t,x) \qquad \forall s,t\in\mathbf{R}, \forall x\in\mathbf{R}^n \tag{1}$$

then it is called a C^1-flow (in $\mathbf{R}^n$). If, in addition, it satisfies

$$\forall t, \qquad \varphi(t,x) \text{ is a polynomial in } x, \tag{2}$$

by which we mean that the components of $\varphi(t,x)$ are polynomials in the components $x_1,\dots,x_n$ of x, then we say that φ is a *polynomial flow.* It is called a *smooth polynomial flow* if it is also C^∞ in t. Polynomial flows were conceived in [6], classified (for $n \le 2$) in [1] (up to polyomorphisms of $\mathbf{R}^2$), and investigated further in [2], [3], [4], [7], and [8]. A diffeomorphism $\not p\colon \mathbf{R}^n \to \mathbf{R}^n$ of $\mathbf{R}^n$ into itself for which both $\not p$ and $\not p^{-1}$ are polynomial maps is called a *polyomorphism* of $\mathbf{R}^n$. If $\varphi(t,x)$ is a polynomial flow in $\mathbf{R}^n$, then it follows from (1) that, for each fixed t, the mapping φ^t defined by $\varphi^t(x) = \varphi(t,x)$ is a polyomorphism of $\mathbf{R}^n$. A function φ satisfying (2) can be regarded as a *path* in the space $\mathscr{P}(\mathbf{R}^n)$ of polynomial maps $\not p\colon \mathbf{R}^n \to \mathbf{R}^n$. The group $\mathrm{GA}(\mathbf{R}^n)$ of all polyomorphisms of $\mathbf{R}^n$ is a subspace of $\mathscr{P}(\mathbf{R}^n)$. Those paths in $\mathscr{P}(\mathbf{R}^n)$ which also satisfy (1) are 1-parameter subgroups of $\mathrm{GA}(\mathbf{R}^n)$. The derivative

$$\dot\varphi(t_0,x) \equiv \frac{\partial\varphi}{\partial t}(t_0,x)$$

could be regarded as the tangent vector to the path φ at the point t_0.

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In [1] it was shown that a C^1 polynomial flow necessarily also satisfies

$$\dot{\varphi}(0,x) \text{ is polynomial in } x \tag{3}$$

That is, a C^1 vector field $V: \mathbf{R}^n \to \mathbf{R}^n$ which has a polynomial flow $\varphi(t,x)$,

$$\dot{\varphi}(t,x) = V(\varphi(t,x)), \qquad \varphi(0,x) = x, \tag{4}$$

is necessarily a *polynomial vector field*: i.e.,

$$V \in \mathscr{P}(\mathbf{R}^n). \tag{5}$$

This may seem obvious and trivial in the light of the obvious identity

$$V(x) \equiv \dot{\varphi}(0,x) \tag{6}$$

which follows from (4) and the property (2) of φ.

However, as the following theorem shows, without the flow hypothesis (1), $\dot{\varphi}(0,x)$ need not even be analytic in x. That is, there are C^∞ functions φ satisfying (2) for which $\dot{\varphi}(0,x)$ is not analytic at any x. That such examples exist was believed by one of us (Meisters) since his paper [6] and earlier. But the discussion of such examples given in [1] is incomplete and inconclusive, while [7, Example 7.1] is only for class C^1 functions φ.

Theorem. *Let f be a C^∞ function on $\mathbf{R}$. Then there is a C^∞ function φ on $\mathbf{R} \times \mathbf{R}$ such that*

1. *$\varphi(t,x)$ is a polynomial in x for each t,*
2. *$\dot{\varphi}(0,x) = f(x)$.*

(*Note that $f(x)$ need not be analytic at any point. See Mandelbrojt* [5].).

Proof. Let g be a C^∞ function which vanishes outside (-1,1), is odd, and has $g'(0) = 1$. Set

$$a_n(t) = g((n+1)t)/(n+1), \qquad n = 0, 1, 2, \ldots.$$

Then the desired C^∞ function $\varphi(t,x)$ will be given by the series

$$\varphi(t,x) \equiv \sum_{n=0}^{\infty} a_n(t) P_n(x) \tag{7}$$

for suitably chosen polynomials $P_n(x)$ which will be formed by approximating $f(x)$ and its derivatives.

First we use the Weierstrass Approximation Theorem to get polynomials $R_n(x)$ such that

$$|f^{(n)}(x) - R_n(x)| < 1/2 \qquad \text{when } |x| \le n.$$

Let $Q_n(x)$ be the polynomial such that $Q_n^{(n)}(x) = R_n(x)$ and $Q_n^{(k)}(0) = f^{(k)}(0)$, $k = 0, 1, \ldots, n$. I.e., $Q_n(x)$ is an n-fold antiderivative of $R_n(x)$.

We will use the following result which follows from Taylor's formula with remainder.

Lemma. *Suppose* $|h^{(n)}(x)| \leq M$ *for* $|x| \leq b$ *and* $h^{(k)}(0) = 0$ *for* $k = 0, 1, 2 \ldots,$ $n-1$. *Then* $|h^{(k)}(x)| \leq M|x|^{n-k}/(n-k)!$ *when* $|x| \leq b$ *and* $k = 0, 1, \ldots, n$.

Applying this lemma to $f(x) - Q_n(x)$, we conclude that

$$|f^{(k)}(x) - Q_n^{(k)}(x)| \leq |x|^{n-k}/2(n-k)! \tag{8}$$

when $|x| \leq n$ and $k = 0, 1, \ldots, n$. Now let $P_0(x) = Q_0(x)$ and

$$P_n(x) = Q_n(x) - Q_{n-1}(x) \qquad \text{for } n \geq 1 \tag{9}$$

From (8) we obtain the following estimates on the derivatives of these polynomials.

$$|P_n^{(k)}(x)| \leq |x|^{n-k}/(n-1-k)! \tag{10}$$

when $|x| \leq n$ and $0 \leq k \leq n-1$.

In order to show that $\varphi(t, x)$, given by (7), is C^∞ it suffices to show that

$$\sum_{n=0}^{\infty} a_n^{(i)}(t) P_n^{(k)}(x)$$

converges uniformly in each strip $|x| \leq b$. Note first that

$$a_n^{(j)}(t) = (n+1)^{j-1} g^{(j)}((n+1)t)$$

and thus $|a_n^{(j)}(t)| \leq (n+1)^j G_j$ where G_j is a bound on $|g^{(j)}|$. Now, from this inequality and (10),

$$|a_n^{(j)}(t) P_n^{(k)}(x)| \leq (n+1)^j G_j b^{n-1-k}/(n-1-k)!$$

when $|x| \leq b, n \geq b$, and $n > k$. The ratio test shows that

$$\sum_n (n+1)^j b^n/(n-1-k)!$$

is convergent.

Thus

$$\frac{\partial^{j+k}}{\partial t^j \partial x^k} \varphi(t, x) = \sum_{n=0}^{\infty} a_n^{(j)}(t) P_n^{(k)}(x),$$

and since the series is a uniformly convergent series of C^∞ functions, $\varphi(t, x)$ is also C^∞ on $\mathbf{R} \times \mathbf{R}$. □

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DEPARTMENT OF MATHEMATICS, KENYON COLLEGE, GAMBIER, OHIO 43022

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEBRASKA–LINCOLN, LINCOLN, NEBRASKA 68588