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WEAKLY COMPACT HOMOMORPHISMS FROM C*-ALGEBRAS ARE OF FINITE RANK

MARTIN MATHIEU

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ABSTRACT. We give a straightforward proof of the fact that every weakly compact homomorphism from a C*-algebra is a finite rank operator.

In [3] Ghahramani proved that every compact homomorphism between C*-algebras is a finite rank operator. Recently, Galé and Ransford [2] obtained the same conclusion under the more general hypothesis of weak compactness by first proving the result for the Wiener algebra and then applying Newburgh's theorem on the continuity of the spectral radius as well as Aupetit's scarcity theorem. The purpose of this note is to present a very simple proof that merely uses some basic facts of functional analysis.

Theorem. *Every weakly compact homomorphism from a C*-algebra into a normed algebra is of finite rank.*

Proof. Let $\pi: A \to B$ be such a homomorphism. Then, $\tilde{A} = A/\ker \pi$ is a C*-algebra and we have a monomorphism $\tilde{\pi}: \tilde{A} \to B$ defined by $\tilde{\pi} \circ \rho = \pi$, where ρ is the canonical quotient map. Since ρ is open, every bounded subset of $\tilde{A}$ is the image of a bounded set in A; thus $\tilde{\pi}$ is weakly compact. Let $x \in A$. By Gelfand theory, the restriction of $\tilde{\pi}$ to the abelian C*-subalgebra generated by $\rho(x^* x)$ is spectral radius preserving. Since the spectral radius coincides with the norm for positive elements, we conclude that

$$\|\rho(x)\|^2 = \|\rho(x)^* \rho(x)\| \leq \|\tilde{\pi}(\rho(x)^* \rho(x))\| \leq \|\tilde{\pi}\| \, \|\rho(x)\| \, \|\tilde{\pi}(\rho(x))\|.$$

Thus, $\tilde{\pi}$ is bounded below. Denoting by $\tilde{\pi}^{-1}: \tilde{\pi}(\tilde{A}) \to \tilde{A}$ a continuous inverse of $\tilde{\pi}$, we infer that $\mathrm{id}_{\tilde{A}} = \tilde{\pi}^{-1} \circ \tilde{\pi}$ is weakly compact whence $\tilde{A}$ is reflexive. But a reflexive C*-algebra is finite dimensional which yields that $\pi(A) = \tilde{\pi}(\tilde{A}) \cong A/\ker \pi$ is finite dimensional. This completes the proof.

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Remark. The fact that every continuous isomorphism from a C*-algebra is bounded below seems to have been noticed first in [1, Lemma 5.3].

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Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-7400 Tübingen, Federal Republic of Germany