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HYPERBOLIC SURFACES AND QUADRATIC EQUATIONS IN GROUPS

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ABSTRACT. A group of a hyperbolic 2-complex K is a group with its associated van Kampen diagrams satisfying a hyperbolic curvature condition and a link condition on the degree of the interior vertices. A *solution* of an equation $W(y_1, \dots, y_n) = 1$ in K, where W is a path in a 2-complex B, is a mapping $\zeta: B \to K$ such that $\zeta W = W(\zeta y_1, \dots, \zeta y_n)$ is contractible in K. This solution ζ is *free* if there is a mapping $h: B \to K^{(1)}$ such that $W(hy_1, \dots, hy_n)$ is contractible in $K^{(1)}$ and such that $\zeta = \pi h$, where π is the projection $\pi: K^{(1)} \to K$. Our main result is that each quadratic equation $W = 1$ has only finitely many nonfree solutions in K. Our tool is essentially the cancellation diagrams on surfaces developed by the present author based on work of Schupp.

1. INTRODUCTION

We will study solutions of quadratic equations in groups. Classification of solutions of such equations in terms of *free* and *nonfree* solutions is first studied by Schupp [10]. There he proves that all solutions of a quadratic equation of nonpositive Euler characteristic are free in groups satisfying certain irregular small cancellation conditions. Consequently the present author [5] improves Schupp's methods and extends his results to broader classes of groups, i.e., small cancellation quotients of free products, of amalgamated products, and of HNN-extensions. Here we shall study another class of groups, the groups of hyperbolic 2-complexes. Such groups have recently been studied by Gersten [3] and Pride [8].

A *group of a hyperbolic 2-complex* is, roughly speaking, a group G that satisfies irregular small cancellation conditions $C(p)$ and $T(q)$ with $(1/p + 1/q) < 1/2$. For example, the $C(3)$ and $T(7)$, or the $C(4)$ and $T(5)$, or the $C(6)$ and $T(4)$ groups are hyperbolic. For a clear discussion on the connection between small cancellation groups and groups of hyperbolic 2-complexes, we refer to Pride [8].

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The main tool we will make use of is the *modified van Kampen diagrams* together with their geometric duals, the *modified pictures*, which are both developed in [5], i.e., the cancellation diagrams that are bounded by nonreduced contractible paths. And here we will use them in the same manner as there.

2. NOTATION AND DEFINITIONS

Let X be a 1-*complex* with the vertex set V and the edge set E. Let $K = \langle X|R \rangle$ be a 2-*complex* where R is the set of defining paths. We shall always assume that the defining paths are cyclically reduced. Also, R is *symmetrized* if all cyclic permutations of $r^{\pm 1} \in R$ also belong to R. In case X is a bouquet with a single vertex, we call K a *presentation.* We shall always use $K = \langle X|R \rangle$ to denote a presentation. Also, we shall denote the 1-skeleton of K by $K^{(1)}$.

Let K, L be two presentations. A mapping of 2-complexes from K to L consists of a mapping of 1-complexes from $K^{(1)}$ to $L^{(1)}$ that takes contractible paths in K to contractible paths in L.

Next, we introduce another 1-complex, written as K^{st}, associated to the 2-complex K. We call K^{st} the *star complex* of K, which is defined as follows.

(1) The vertex set of K^{st} is the set E.

(2) The edge set is defined on R: if $e_1 e_2$ is a subpath in a path r_i of R, then K^{st} has an edge from e_1 to e_2^{-1}.

Now we are ready to introduce hyperbolic 2-complexes. First, we define a *weight function* m to be a mapping from the edge set E of K into the real numbers: $m: E \to \mathbf{R}$, with $m(e^{-1}) = m(e)$ for all e in E. If $\gamma = e_1 e_2 \cdots e_n$ is a defining path in K, then we define $m(\gamma) = \sum_{i=1}^{n} m(e_i)$. Then, the interesting situation to us is that when we have the 2-complex K together with a weight function m defined on K^{st}. We denote this situation by (K, m). Then we define the weight function m^* on K^{st} as follows: If $\gamma = e_1 e_2 \cdots e_n$ is a defining path, then $m^*(\gamma) = m(\alpha_{e_1 \wedge e_2}) + (\alpha_{e_2 \wedge e_3}) + \cdots + m(\alpha_{e_{n-1}} \wedge e_n) + m(\alpha_{e_n} \wedge e_1)$, where $\alpha_{e_i \wedge e_{i+1}}$ is the oriented edge of K^{st} from e_i to e_{i+1} (see Figure 1).

Following Gersten, we call (K, m) *hyperbolic* if the following hold:

(i) *Link condition.* If s is a reduced nonempty closed path in K^{st}, then $m^*(s) \geq 2$.

(ii) *Curvature condition.* For each $r \in R$, $m^*(r) < L(r) - 2$, where $L(r)$ is the length of r.

These conditions, as will be seen in §3, have clear geometric meanings on cancellation diagrams.

In passing, we mention that, in obtaining a combinatorial proof of Riemann-Hurwitz formula, Lyndon [6] used conditions similar to the above.

We now consider another presentation. Let $B = \langle Y|W \rangle$ be a presentation where Y is a 1-complex (disjoint from X) and W is the only defining path. Then W is called *quadratic* if each edge of Y occurring in W occurs exactly twice in W. In general, a quadratic path W defines a closed surface S as the

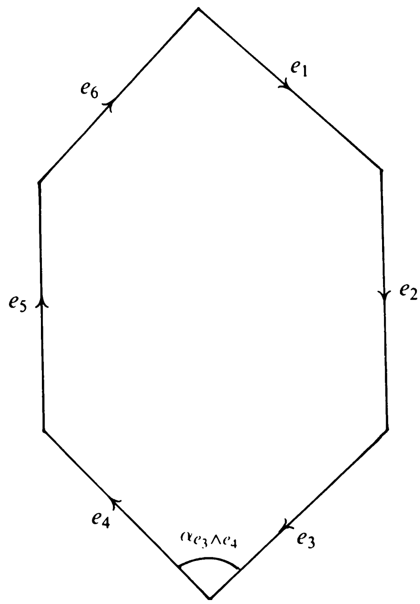

FIGURE 1.

quotient space of a polygonal region whose sides are labelled, in counterclockwise order, by the edges of W, with each pair of directed sides bearing the same label identified. This fact plays a crucial role in our work.

We say that W is *orientable* (*nonorientable*) if the surface S defined by W is orientable (nonorientable). Then W is *orientable* only if every edge of Y occurring in W occurs exactly twice, with opposite signs. Otherwise W is *nonorientable.* We define the Euler characteristic of E to be $\chi(W) = V - E + F$, the Euler characteristic of the surface defined by W, where V, E and F are the number of vertices, edges and regions of the surface.

Definition 2.1. A *solution* of $W(y_1, \dots, y_n) = 1$ in K is a mapping $\zeta \colon B \to K$ such that $\zeta W = (\zeta y_1, \dots, \zeta y_n)$ is contractible in K.

Definition 2.2. This solution ζ is *free* if there is a mapping $h \colon B \to K^{(1)}$ such that $W(hy_1, \dots, hy_n)$ is contractible in $K^{(1)}$ and such that $\zeta = \pi h$, where π is the projection $\pi \colon K^{(1)} \to K$.

The following are our main theorems, which we view as a supplement to the results for small cancellation groups [5].

Theorem B. *Let $W(y_1, \dots, y_n)$ be a quadratic path with $\chi(W) \geq 0$ in $B = \langle Y | W \rangle$, and let $K = \langle X | R \rangle$ be hyperbolic where R is symmetrized. Assume that R contains no proper powers if W is nonorientable. Then all solutions ζW of the equation $W(y_1, \dots, y_n) = 1$ in K are free.*

Theorem C. *Let $W(y_1, \ldots, y_n)$ be a quadratic path with $\chi(W) < 0$ in $B = \langle Y|W\rangle$, and let $K = \langle X|R\rangle$ be hyperbolic where R is symmetrized. Assume that R contains no proper powers if W is nonorientable. Then there exist only finitely many nonfree solutions of the equation $W(y_1, \ldots, y_n)$ in K.*

3. Cancellation diagrams

Here we briefly state the definitions of a van Kampen diagram M and its dual, the picture P. For more information, see [5].

Definition 3.1. A *van Kampen diagram* M over a presentation $K = \langle X|R\rangle$ on a compact surface S is a tessellation M of S with each region except one in M having its boundary one of the defining paths of R, and the boundary path W of the exceptional region D_∞ is contractible in K.

Definition 3.2. Call M a *modified van Kampen diagram*, written as M^*, if we don't require that W be reduced.

We say that M is not *reduced* if M contains two regions D_1 and D_2 (not necessarily distinct) such that α is a common path of $D_1 \cup D_2$ and the boundary path of $\{\partial D_1 \cup \partial D_2 - \alpha\}$ is a trivial path in K (see Figure 2). To reduce M, we press together D_1 and D_2.

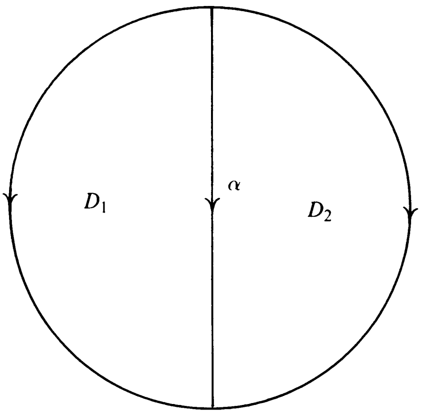

FIGURE 2.

Proposition 3.1 [5, Proposition 2.1]. *A modified van Kampen diagram M^* is obtainable from a reduced van Kampen diagram M by attaching trees inside D_∞ to the boundary ∂D_∞.*

Outline of proof. Let $W(x_1, \ldots, x_n)$ be a contractible path in K. Let M_0 be the "bouquet of balloons" given by $W(x_1, \ldots, x_n) = \prod_{i=1}^n u_i r_i u_i^{-1}$, and let the unreduced path W_0 be the boundary label of M_0. Let $W_1 = x_{i_1} x_{i_2} \cdots x_{i_m}$ be the reduced form of W_0. Then this W_1 bounds a reduced van Kampen diagram M_1. We can pass from W_1 to W_0 by a succession of insertions of parts xx^{-1}. We modify M_1 to M_0 with boundary label W_1 by attaching, exterior to M_1, trees with each spur labelled xx^{-1}, i.e., to pass from ab to $axx^{-1}b$ (see Figure 3). □

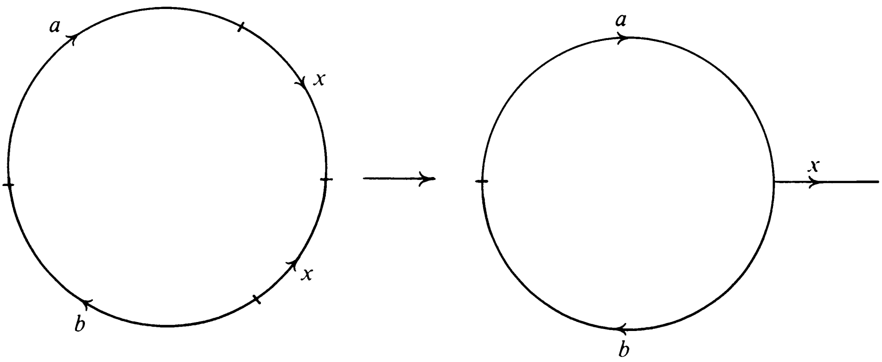

FIGURE 3.

Next, we consider the duals of M. We define a *picture* P over $K = \langle X|R\rangle$ to be the geometric dual of a van Kampen diagram M over K. Also a *modified picture* P^* is the geometric dual of a corresponding modified van Kampen diagram M^*.

A picture P is not *reduced* if P contains two small discs v_1^* and v_2^* such that the paths around the arcs on v_1^* and v_2^* beginning at the end points of a common arc e^* are mutually inverse paths in K (see Figure 4).

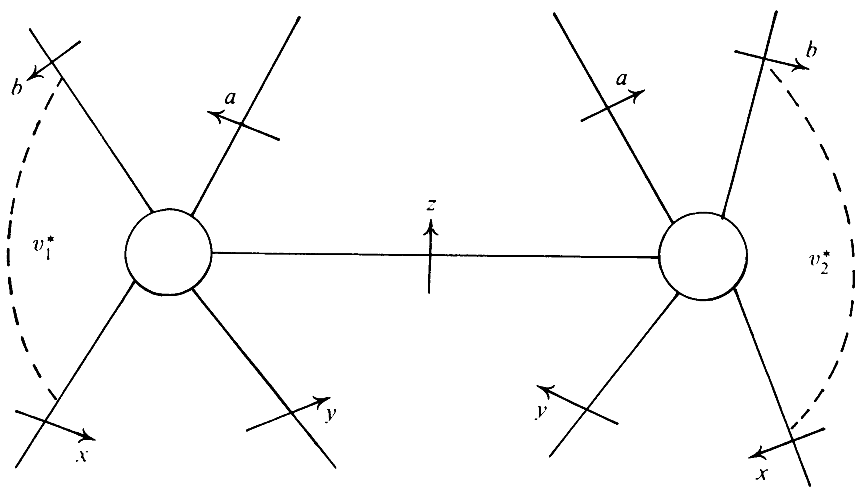

FIGURE 4.

To reduce P, we will use the so called bridge moves. A *bridge move* is the following move between two arcs with the same label but oppositely oriented (see Figure 5).

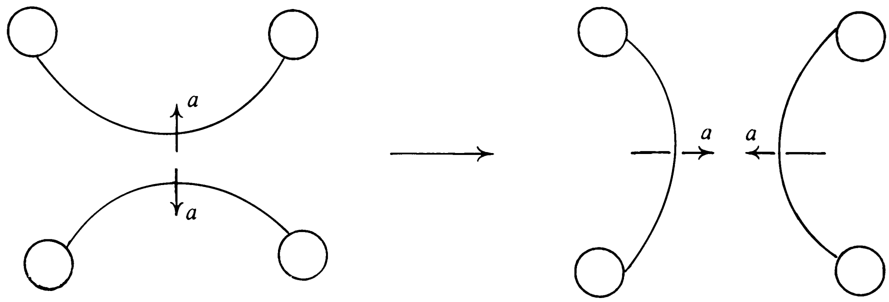

FIGURE 5.

Now on a cancellation diagram M, the meanings of the link and curvature conditions are clear. We see that $m^*(r)$ is the sum of the weights on interior corners of a region D with boundary path r. We state these conditions on M as follows.

(i) *Link condition.* For each reduced nonempty closed path s in the star complex of the vertices of M, we have $m^*(s) \geq 2$.

(ii) *Curvature condition.* For each region D of M, $m^*(D) < L(D) - 2$, where $L(D)$ is the number of boundary edges of D.

Now on a picture P, these two conditions reverse.

(i)* *Link condition.* For a small disc v^* of P with arcs $e_1^*, \dots, e_n^*$ attached, we have $m^*(v^*) < L(v^*) - 2$, where $m^*(v^*)$ is the sum of the weights on oriented corners between e_i^* and e_{i+1}^* on v^*, and $L(v^*)$ is the number of arcs attached on v^*.

(ii)* *Curvature condition.* For each region Δ of P, $m^*(\Delta) \geq 2$.

Definition 3.3. A tessellated surface S^T is called *hyperbolic* if S^T satisfies the link and curvature conditions.

4. MAIN RESULTS

First, we have the following theorem that establishes a relation between a nonfree solution and a reduced nonempty tessellated surface S^T.

Theorem A [5, Theorem 3.1]. *Let $W(y_1, \dots, y_n)$ be a quadratic path in $B = \langle Y|W\rangle$, and let $K = \langle X|R\rangle$ where R is symmetrized. Assume that R contains no proper powers if W is nonorientable. If K admits a non-free solution ζW of the equation $W(y_1, \dots, y_n) = 1$, then there exists a reduced nonempty tessellated surface S^T defined by ζW.*

Outline of proof. Since K admits a nonfree solution ζW, by Proposition 3.1 we have a modified van Kampen diagram M^* defined by ζW. Now we transform M^* to its dual, the modified picture P^*. Then, to obtain a closed surface, we identify the pairing arcs underlying the boundary ∂P^*. Next, we carry out reductions on this surface, hence obtaining a reduced nonempty tessellated surface, still denoted by S^T. □

Now, to lift this surface S^T up to a hyperbolic surface, written as S^H, we assume on S^T the following conditions.

(i)* For each region Δ of S^T, $m^*(\Delta) \geq 2$.

(ii)* For each vertex v^* in S^T, $m^*(v^*) < L(v^*) - 2$, where $L(v^*)$ is the number of arcs around v^*.

Next, we have a crucial lemma.

Lemma 4.1. *On the hyperbolic surface S^H, we have*

$$V \leq -\chi(W)/\varepsilon,$$

where V is the number of vertices of S^H, $\chi(S^H)$ is Euler characteristic of S^H, and $\varepsilon = \mathrm{Minimum}\{L(v^*) - m^*(v^*) - 2\}$.

Proof. In the following, all the summations are all over the entire S^H.

First we have

$$\begin{aligned} \sum_{v^*} m^*(v^*) &\leq \sum_{v^*}[L(v^*) - (2+\varepsilon)] \\ &\leq 2E - (2+\varepsilon)V \end{aligned} \tag{1}$$

where we count the number of arcs with multiplicity. Second, we have

$$\sum_{\Delta} m^*(\Delta) \geq 2F. \tag{2}$$

Thus, combining (1) and (2) we have

$$2F \leq 2E - (2+\varepsilon)V,$$

that is,

$$2\varepsilon V \leq -2(V - E + F) = -2\chi(S^H).$$

Hence

$$V \leq -\chi(S^H)/\varepsilon. \tag{3}$$

Since the process of reductions of cancellation diagrams may decrease the genus of a surface, we have $\chi(W) \leq \chi(S^H)$, hence $-\chi(W) \geq -\chi(S^H)$. Therefore, from (3) we have

$$V \leq \chi(W)/\varepsilon. \quad \square$$

Now readily we have the following two theorems.

Theorem B. *Let $W(y_1, \dots, y_n)$ be a quadratic path with $\chi(W) \geq 0$ in $B = \langle Y|W\rangle$, and let $K = \langle X|R\rangle$ be hyperbolic where R is symmetrized. Assume that R contains no proper powers if W is nonorientable. Then all solutions ζW of the equation $W(y_1, \dots, y_n) = 1$ in K are free.*

Proof. If $\chi(W) \geq 0$, then Lemma 4.1 assures that the number of vertices in S^H is zero, hence no such surface S^H can occur. Therefore, we conclude that all solutions are free. □

Theorem C. *Let $W(y_1, \ldots, y_n)$ be a quadratic path with $\chi(W) < 0$ in $B = \langle Y|W\rangle$, and let $K = \langle X|R\rangle$ be hyperbolic where R is symmetrized. Assume that R contains no proper powers if W is nonorientable. Then there exist only finitely many nonfree solutions of the equation $W(y_1, \ldots, y_n) = 1$ in K.*

Proof. If $\chi(W) < 0$, then Lemma 4.1 implies that the number of vertices of S^H is bounded. Then for each surface S^H with a fixed number of vertices, there exist only finite number of triangulated 2-complexes B (including different labellings). Hence there are only finitely many nonfree solutions. □

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