LENS SPACES AND DEHN SURGERY

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Abstract. The question of when a lens space arises by Dehn surgery is discussed with a characterization given for satellite knots. The lens space $L(2, 1)$, i.e. real projective 3-space, is shown to be unobtainable by surgery on a symmetric knot.

The problem of when a lens space can be obtained by performing Dehn surgery on a knot in the 3-sphere has been of interest to topologists for some time. It is known that certain lens spaces can arise by Dehn surgery on torus knots (Moser [Mo]), certain pretzel knots (Fintushel-Stern [FS]), and certain nontrivial satellite knots (in fact, certain cables of torus knots, Bailey-Rolfsen [BR]).

In this note we show how some recent developments in 3-manifold theory shed more light on this problem. In Theorem 1 we use recent results of Culler-Gordon-Luecke-Shalen [CGLS], Gabai [Ga], Gordon [Go], and Scharlemann [S] to characterize how a lens space can be obtained by surgery on a (nontrivial) satellite knot. Theorem 1 was originally proven by Wu [Wu]. We present here a somewhat more concise proof, discovered independently, which makes deeper use of the critical theorems of Gabai and Gordon. Similar results have also been obtained by Wang [W2] and Hempel [H]. We then specialize to the question of when real projective 3-space, i.e. the lens space $L(\pm 2, 1)$, can be obtained by Dehn surgery. Using results of Thompson [T] and Wang [W2], we show in Theorem 2 that no surgery on a nontrivial symmetric knot yields this manifold.

Theorem 1. If nontrivial Dehn surgery on a satellite knot yields a manifold with cyclic fundamental group, then the knot is a cable of a torus knot and the knot and surgery coefficient are as in [Go, Theorem 7.5 (iii), $k = 2$]. I.e. the knot is the $(2pq \pm 1, 2)$-cable on a $(p, q)$-torus knot, the surgery coefficient is $4pq \pm 1$, and the resulting manifold is $L(4pq \pm 1, 4q^2)$.

Corollary. If a lens space $L$ is obtained by Dehn surgery on a satellite knot then $|\pi_1(L)| \geq 23$. 

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Proof of Theorem 1. Our notation will mostly follow that of [Go]. Let $K$ be a nontrivial knot in the 3-sphere $S^3$, let $J$ be a knot nontrivially contained in $S^1 \times D^2$ with winding number $w \geq 0$, and let $J(K)$ be the satellite of $K$ determined by $J \subseteq S^1 \times D^2$. Further let $(J;r)$ be the manifold obtained from $S^1 \times D^2$ by $r$-surgery on $J$, $r \neq 1/0$; and let $(J(K);r)$ be the manifold obtained from $S^3$ by $r$-surgery on $J(K)$. Since $J(K)$ is not a torus knot, Corollary 1 of [CGLS] implies that $(J(K);r)$ does not have cyclic fundamental group unless $r$ is an integer; we therefore consider only surgery with an integral coefficient $m$ (so that our notation agrees with that of [Go]).

We begin by reviewing what is known when $(J;m)$ has compressible boundary.

1. By the proof of [S, Corollary 5.2], $w \neq 0$.

2. The manifold $(J;m)$ is homeomorphic to $V\#M$ where $V \cong S^1 \times D^2$ and $M$ is some closed 3-manifold. Denoting the g.c.d. of $w$ and $m$ by $(w,m)$, we have [by Go, Lemma 3.3] that $H_1(M) \cong \mathbb{Z}_{(w,m)}$, and a meridian of $V$ has slope $m/w$ relative to the original $S^1 \times D^2$. Moreover, by the proof of [Go, Lemma 3.6], $w$ and $m/(w,m)$ are relatively prime. It follows that $m/w$ has denominator at least $w$, and is therefore not an integer unless $w = 1$.

3. By [Ga, Theorem 1.1], either $H_1(M) \neq 0$ (i.e. $(w,m) \neq 1$) or $(J;m)$ is homeomorphic to $S^1 \times D^2$ and $J$ is a 0- or 1-bridge braid. If $w = 1$, then we must have the second possibility; but this implies that $J$ is a core of $S^1 \times D^2$. Hence $w \geq 2$.

Now suppose that $(J(K);m)$ has cyclic fundamental group for some integer $m$. Then $(J;m)$ has compressible boundary and all of the above applies. It follows that $(J(K);m) \cong (K;m/w)^2\#M$. Hence $(K;m/w^2)$ has cyclic fundamental group, and since $m/w^2$ is not an integer, Corollary 1 of [CGLS] implies that $K$ is a torus knot. Hence the fundamental group of $(K;m/w^2)$ is nontrivial, and so the fundamental group of $M$ must be trivial. By (3) above, this implies that $(J;m)$ is homeomorphic to $S^1 \times D^2$ and $J$ is a 0- or 1-bridge braid. A 0-bridge braid is a cable, so it only remains to eliminate the possibility that $J$ is a 1-bridge braid.

Suppose that $J$ is a 1-bridge braid. In §3 of [Ga] there are associated to $J$ two integers, the braid width $b(1 \leq b \leq w - 2)$ and $t(1 \leq t \leq w - 1)$. By [Ga, Lemma 3.5] and the remarks just before [Ga, Definition 3.3], the surgery coefficient $m$ is equal to $\pm(wt + d)$ depending on the orientation convention, where $d$ is either $b$ or $b + 1$. On the other hand, as $K$ is a torus knot and as $\pi_1(K;m/w^2)$ is cyclic, [Go, Corollary 7.4] implies that $m$ is congruent to $\pm 1 \mod w^2$. (Note that $w$ and $m$ are relatively prime since $H_1(M) = 0$.) Hence $d$ is congruent to $\pm 1 \mod w$. Now $b$ is equal to either $d$ or $d - 1$ and $1 \leq b \leq w - 2$, so either $b = 1$ or $b = w - 2$. Replacing $J$ by its mirror image replaces $b$ by $w - b - 1$ (see the remark after [Ga, Lemma 3.4]), so it
is enough to consider the case $b = 1$. But this means that $J$ is a $(2,1)$-cable of a cable (see the remark after [Ga, Examples 3.8]), so that $J(K)$ is a cable of a cable of a torus knot. This contradicts [Go, Theorem 7.5]. □

**Theorem 2.** Real projective 3-space cannot be obtained by Dehn surgery on a nontrivial symmetric knot $K$ in the 3-sphere.

**Proof of Theorem 2.** Recall that a knot $K$ in $S^3$ is symmetric if it is invariant under the action on $S^3$ of some nontrivial finite group $G$. Without loss of generality we may take $G$ to be a cyclic group $\mathbb{Z}_n$. From Moser [Mo] we conclude that $K$ is not a torus knot. We then reduce to the strongly invertible case by appealing to Wang [Wa]. He proves that if $K$ admits an action of a cyclic group $\mathbb{Z}_n$ and is not a torus knot, then if $n > 2$ or if $n = 2$ and the action is fixed point free, no nontrivial surgery on $K$ yields a lens space. He also shows that if $n = 2$ and $K$ is disjoint from the fixed point set, then no nontrivial surgery can yield real projective 3-space. The only symmetry left to consider is a strong inversion.

So suppose that surgery on some nontrivial strongly invertible knot $K$ gives $RP^3$. As before, we begin by recalling some general facts about surgery on strongly invertible knots. The strong inversion on $K$ extends to an involution on each of the manifolds $(K;r)$ obtained from $S^3$ by performing $r$-fold surgery on $K$. For each $(K;r)$ the quotient under this involution is the 3-sphere and hence each $(K;r)$ double branch covers $S^3$. Moreover, the branch set of this covering can be obtained by removing a trivial tangle from the unknot (i.e. the image of the axis of the strong inversion) and replacing it in a manner determined by the surgery coefficient $r$. See, for example, Montesinos [M] or, for a more explicit construction, [B].

As $K$ is not a torus knot, we can apply the cyclic surgery theorem of [CGLS] to conclude that the surgery coefficient $r$ is at a distance 1 from the meridian of $K$. It follows that the removal and replacement of the trivial tangle corresponding to $r$ is in fact the attachment of a band to the unknot. The core of this band is the image of our original knot $K$ under the quotient map $(K;1/0) \to S^3$; again see [M] or [B].

By Hodgson and Rubenstein [HR], lens spaces uniquely double branch cover the 3-sphere with branch set the appropriate two-bridge knot or link. For real projective space this two-bridge link is the Hopf link. We conclude that obtaining $RP^3$ by surgery on $K$ corresponds in the quotient $S^3$ to obtaining the Hopf link by attaching a band to the unknot.

Now we apply a theorem of Thompson [T, Corollary 3] which states that there is a unique band which creates the unknot from the Hopf link, or, dually, that there is a unique band which creates the Hopf link from the unknot. It follows that we have one of the two pictures in Figure 1, and that our original knot $K$ is unknotted.
We close with some questions and conjectures.

1. Is it possible to obtain a lens space $L$ with $|\pi_1(L)| < 5$ by Dehn surgery on a nontrivial knot? Conjecture: No.
2. Is it possible to obtain a lens space $L$ with $|\pi_1(L)| < 18$ by Dehn surgery on a nontorus knot? Conjecture: No.

References


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