A SIMPLE PROOF OF LIVINGSTON'S INEQUALITY FOR CARATHÉODORY FUNCTIONS

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Abstract. The Livingston determinant inequality involving the Maclaurin co-
efficients of a Carathéodory function are derived in a straightforward manner
by use of the Riesz-Herglotz representation and the Schwarz inequality. The
result is extended to the case of matrix-valued functions.

1. Introduction

Let \( f(z) = \sum_{k=0}^{\infty} f_k z^k \) be a function of the complex variable \( z \), analytic in
the unit disc \( D = \{ z : |z| < 1 \} \) and satisfying

\[ \text{Re}(f(z)) \geq 0 \quad \text{for } z \in D. \]

Then \( f(z) \) is called a Carathéodory function. The theory of this class of func-
tions has numerous applications in pure and applied mathematics. An intro-
duction to the subject can be found in Akhiezer's classical book [1]. In the
sequel it is assumed that \( f(z) \) does not reduce to an imaginary constant or,
equivalently, that the real part of \( f_0 \) is strictly positive.

Consider the reciprocal function \( g(z) = f(z)^{-1} = \sum_{k=0}^{\infty} g_k z^k \). (Note that
\( g(z) \) is a Carathéodory function.) For any nonnegative integer \( n \), let \( g_n(z) = \sum_{k=0}^{n} g_k z^k \) denote the order \( n \) truncation polynomial of \( g(z) \). Define the
function \( v_n(z) = \sum_{k=0}^{\infty} v_{n,k} z^k \), analytic at the origin, through the identity

\[ g_n(z)f(z) = 1 + 2z^{n+1}v_n(z). \]

Motivated by a coefficient problem for a certain class of multivalent functions,
Livingston has recently established the following remarkable inequality [4]:

\[ |v_{n,k}| \leq |f_0|^{-1} \text{Re}(f_0), \]

for all nonnegative integers \( n \) and \( k \).

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representation.
As shown in [4], the number $v_{n,k}$ can be expressed in terms of the determinant of the Hessenberg matrix

$$F_{n,k} = \begin{bmatrix}
    f_1 & f_2 & \cdots & f_n & f_{n+k+1} \\
    f_0 & f_1 & \cdots & f_{n-1} & f_{n+k} \\
    f_0 & f_1 & \cdots & f_{n-2} & f_{n+k-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    f_0 & f_1 & \cdots & f_0 & f_{k+1}
\end{bmatrix}$$

built from the Maclaurin coefficients $f_m$ of the given function $f(z)$. In fact, it is easily seen that $v_{n,k}$ is given by

$$v_{n,k} = (-1)^n \frac{\det(F_{n,k})}{2 f_0^{n+1}}.$$

This follows from the fact that (2) contains the system of linear relations

$$[g_0, g_1, \ldots, g_{n-1}, g_n] F_{n,k} = [0, 0, \ldots, 0, 2v_{n,k}].$$

Solving (6) for the first "unknown", $g_0$, we obtain (5) by use of $g_0 = f_0^{-1}$. Livingston's argument starts from the explicit formula (5), uses a suitable approximation of $f(z)$ by a rational singular Carathéodory function, involves some manipulations of Hessenberg determinants, and produces the result (3) by means of the Cauchy inequality [4].

This paper gives a more direct proof of Livingston's result (3), based on the integral representation of $v_{n,k}$ that follows from the Riesz-Herglotz formula for the Carathéodory function $f(z)$. In this setting, the Schwarz inequality leads immediately to the desired result. After explaining the proof in some detail, we briefly examine the case where the bounds (3) are sharp. A final section contains an extension of Livingston's result to matrix-valued Carathéodory functions.

2. Proof

Let us denote by $d\sigma$ the positive measure that corresponds to the Carathéodory function $f(z)$ via the Riesz-Herglotz integral representation [1], [3]. The relation is given by

$$f(z) = i \text{Im}(f_0) + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta).$$

Roughly speaking, the Maclaurin coefficients of $f(z)$ are the trigonometric moments of $d\sigma$. More precisely, (7) is equivalent to the set of formulas

$$\text{Re}(f_0) = \int_0^{2\pi} d\sigma(\theta),$$

$$f_m = 2 \int_0^{2\pi} e^{-im\theta} d\sigma(\theta) \quad \text{for } 1 \leq m < \infty.$$
The Livingston number \( v_{n,k} \) defined from (2) can be expressed in the form

\[
v_{n,k} = \frac{1}{2} \sum_{l=0}^{n} g_l f_{n+k+1-l}.
\]

(This is the last component of the system (6).) Using (9) we then obtain the key formula

\[
v_{n,k} = \int_{0}^{2\pi} e^{-i(n+k+1)\theta} g_n(e^{i\theta}) d\sigma(\theta).
\]

Applying the Schwarz inequality yields

\[
|v_{n,k}|^2 \leq \int_{0}^{2\pi} d\sigma(\theta) \int_{0}^{2\pi} |g_n(e^{i\theta})|^2 d\sigma(\theta).
\]

Elementary computation shows that (12) amounts to the desired result (3), written in the form

\[
|v_{n,k}|^2 \leq \text{Re}(f_0) \text{Re}(f_0^{-1}).
\]

Indeed, in view of (8) and (9) we can write the identity

\[
\int_{0}^{2\pi} |g_n(e^{i\theta})|^2 d\sigma(\theta) = \frac{1}{2} g_n(F_n + F_n^*) \overline{g_n},
\]

with \( g_n = [g_0, g_1, \ldots, g_n] \) and

\[
F_n = \begin{bmatrix}
  f_0 & f_1 & \cdots & f_n \\
  f_0 & \cdots & f_{n-1} \\
  \vdots & \ddots & \ddots & \vdots \\
  f_0 & \cdots & f_{n-1} & f_0
\end{bmatrix}.
\]

Since \( g_n F_n = [1, 0, \ldots, 0] \), by (2), the right hand side of (14) equals \( \text{Re}(g_0) \). This proves the claim (13), by use of (8) and of \( g_0 = f_0^{-1} \).

The argument above allows us to discuss the "case of equality" in an efficient manner. Indeed, it is seen that (3) is sharp, for given values of \( n \) and \( k \), if and only if the polynomial \( g_n(z) \) is proportional to the monomial \( z^{n+k+1} \) on the support of the measure \( d\sigma \) (viewed as a subset of the unit circle \( |z| = 1 \)). Without going into details, let us mention the following consequence of this fact. For a given \( n \), equality holds in (3) for two successive values of \( k \) if and only if the support of \( d\sigma \) reduces to a single point; this means that \( f(z) \) is a singular Carathéodory function of degree 1, i.e., a function of the form

\[
f(z) = i\beta + \alpha \frac{e^{i\gamma} + z}{e^{i\gamma} - z},
\]

where \( \alpha, \beta \) and \( \gamma \) are real numbers, with \( \alpha > 0 \). In this case, the bound is sharp for all nonnegative integers \( n \) and \( k \).
3. Generalization

Let us now briefly examine how the Livingston inequality can be generalized to the case of a matrix-valued Carathéodory function. The definition is formally the same as in §1, except that the coefficients $f_k$ are square matrices of any fixed order $p$. The condition (1) means that the Hermitian part (or “real part”) of $f(z)$ is nonnegative definite in the unit disc. In addition, we assume here that $f(z)$ is nondegenerate, in the sense that the matrix $\text{Re}(f_0)$ is invertible.

Define the $p \times p$ matrix polynomial $g_n(z)$, of formal degree $n$, and the $p \times p$ matrix function $v_n(z)$ from the identity (2), where $1$ is interpreted as the identity matrix. It can be shown that the matrix coefficients $v_{n,k}$ of $v_n(z)$ satisfy the inequality

\[(17) \quad v_{n,k}v_{n,k}^* \leq \text{Re}(\text{tr}(f_0))\text{Re}(f_0^{-1}),\]

generalizing (15). Here, the star denotes the conjugate transpose, $\text{Re}$ is the Hermitian part, $\text{tr}$ stands for the trace (i.e., the sum of diagonal entries), and the inequality $a \leq b$ means that $b - a$ is nonnegative definite.

The proof of (17) is essentially the same as in §2. It is based on the Riesz-Herglotz representation (7) for nondegenerate matrix-valued Carathéodory functions and makes use of a suitable matrix extension of the Schwarz inequality [2]. Details will not be given.

References