A SIMPLE PROOF OF LIVINGSTON'S INEQUALITY FOR CARATHÉODORY FUNCTIONS

PHILIPPE DELSARTE AND YVES GENIN

(Communicated by Irwin Kra)

Abstract. The Livingston determinant inequality involving the Maclaurin coefficients of a Carathéodory function are derived in a straightforward manner by use of the Riesz-Herglotz representation and the Schwarz inequality. The result is extended to the case of matrix-valued functions.

1. Introduction

Let \( f(z) = \sum_{k=0}^{\infty} f_k z^k \) be a function of the complex variable \( z \), analytic in the unit disc \( D = \{ z : |z| < 1 \} \) and satisfying

\[
\text{Re}(f(z)) \geq 0 \quad \text{for } z \in D.
\]

Then \( f(z) \) is called a Carathéodory function. The theory of this class of functions has numerous applications in pure and applied mathematics. An introduction to the subject can be found in Akhiezer's classical book [1]. In the sequel it is assumed that \( f(z) \) does not reduce to an imaginary constant or, equivalently, that the real part of \( f_0 \) is strictly positive.

Consider the reciprocal function \( g(z) = f(z)^{-1} = \sum_{k=0}^{\infty} g_k z^k \). (Note that \( g(z) \) is a Carathéodory function.) For any nonnegative integer \( n \), let \( g_n(z) = \sum_{k=0}^{n} g_k z^k \) denote the order \( n \) truncation polynomial of \( g(z) \). Define the function \( v_n(z) = \sum_{k=0}^{\infty} v_{n,k} z^k \), analytic at the origin, through the identity

\[
g_n(z)f(z) = 1 + 2z^{n+1}v_n(z).
\]

Motivated by a coefficient problem for a certain class of multivalent functions, Livingston has recently established the following remarkable inequality [4]:

\[
|v_{n,k}| \leq |f_0|^{-1} \text{Re}(f_0),
\]

for all nonnegative integers \( n \) and \( k \).

Received by the editors March 21, 1988 and, in revised form, February 27, 1989.

1980 Mathematics Subject Classification (1985 Revision). Primary 30D50; Secondary 30E05.

Key words and phrases. Carathéodory functions, coefficient inequalities, Riesz-Herglotz representation.
As shown in [4], the number \( v_{n,k} \) can be expressed in terms of the determinant of the Hessenberg matrix

\[
F_{n,k} = \begin{bmatrix}
    f_1 & f_2 & \cdots & f_n & f_{n+k+1} \\
    f_0 & f_1 & \cdots & f_{n-1} & f_{n+k} \\
    f_0 & \cdots & f_{n-2} & f_{n+k-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    f_0 & \cdots & \cdots & f_{k+1}
\end{bmatrix}
\]

built from the Maclaurin coefficients \( f_m \) of the given function \( f(z) \). In fact, it is easily seen that \( v_{n,k} \) is given by

\[
v_{n,k} = (-1)^n \frac{\det(F_{n,k})}{2f_0^{n+1}}.
\]

This follows from the fact that (2) contains the system of linear relations

\[
[g_0, g_1, \ldots, g_{n-1}, g_n]F_{n,k} = [0, 0, \ldots, 0, 2v_{n,k}].
\]

Solving (6) for the first “unknown”, \( g_0 \), we obtain (5) by use of \( g_0 = f_0^{-1} \).

Livingston’s argument starts from the explicit formula (5), uses a suitable approximation of \( f(z) \) by a rational singular Carathéodory function, involves some manipulations of Hessenberg determinants, and produces the result (3) by means of the Cauchy inequality [4].

This paper gives a more direct proof of Livingston’s result (3), based on the integral representation of \( v_{n,k} \) that follows from the Riesz-Herglotz formula for the Carathéodory function \( f(z) \). In this setting, the Schwarz inequality leads immediately to the desired result. After explaining the proof in some detail, we briefly examine the case where the bounds (3) are sharp. A final section contains an extension of Livingston’s result to matrix-valued Carathéodory functions.

2. Proof

Let us denote by \( d\sigma \) the positive measure that corresponds to the Carathéodory function \( f(z) \) via the Riesz-Herglotz integral representation [1], [3]. The relation is given by

\[
f(z) = i\text{Im}(f_0) + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\sigma(\theta).
\]

Roughly speaking, the Maclaurin coefficients of \( f(z) \) are the trigonometric moments of \( d\sigma \). More precisely, (7) is equivalent to the set of formulas

\[
\text{Re}(f_0) = \int_0^{2\pi} d\sigma(\theta),
\]

\[
f_m = 2\int_0^{2\pi} e^{-im\theta} \, d\sigma(\theta) \quad \text{for } 1 \leq m < \infty.
\]
The Livingston number $v_{n,k}$ defined from (2) can be expressed in the form

$$v_{n,k} = \frac{1}{2} \sum_{l=0}^{n} g_l f_{n+k+1-l}.$$  

(This is the last component of the system (6).) Using (9) we then obtain the key formula

$$v_{n,k} = \int_{0}^{2\pi} e^{-i(n+k+1)\theta} g_n(e^{i\theta}) d\sigma(\theta).$$

Applying the Schwarz inequality yields

$$|v_{n,k}|^2 \leq \int_{0}^{2\pi} d\sigma(\theta) \int_{0}^{2\pi} |g_n(e^{i\theta})|^2 d\sigma(\theta).$$

Elementary computation shows that (12) amounts to the desired result (3), written in the form

$$|v_{n,k}|^2 \leq \text{Re}(f_0) \text{Re}(f_0^{-1}).$$

Indeed, in view of (8) and (9) we can write the identity

$$\int_{0}^{2\pi} |g_n(e^{i\theta})|^2 d\sigma(\theta) = \frac{1}{2} g_n(F_n + F_n^*) g_n^*,$$

with $g_n = [g_0, g_1, \ldots, g_n]$ and

$$F_n = \begin{bmatrix} f_0 & f_1 & \cdots & f_n \\ f_0 & f_1 & \cdots & f_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ f_0 & \cdots & \cdots & f_0 \end{bmatrix}.$$ 

Since $g_n F_n = [1, 0, \ldots, 0]$, by (2), the right hand side of (14) equals $\text{Re}(g_0)$. This proves the claim (13), by use of (8) and of $g_0 = f_0^{-1}$.

The argument above allows us to discuss the “case of equality” in an efficient manner. Indeed, it is seen that (3) is sharp, for given values of $n$ and $k$, if and only if the polynomial $g_n(z)$ is proportional to the monomial $z^{n+k+1}$ on the support of the measure $d\sigma$ (viewed as a subset of the unit circle $|z| = 1$). Without going into details, let us mention the following consequence of this fact. For a given $n$, equality holds in (3) for two successive values of $k$ if and only if the support of $d\sigma$ reduces to a single point; this means that $f(z)$ is a singular Carathéodory function of degree 1, i.e., a function of the form

$$f(z) = i\beta + \alpha \frac{e^{i\gamma} + z}{e^{i\gamma} - z},$$

where $\alpha$, $\beta$ and $\gamma$ are real numbers, with $\alpha > 0$. In this case, the bound is sharp for all nonnegative integers $n$ and $k$. 

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3. Generalization

Let us now briefly examine how the Livingston inequality can be generalized to the case of a matrix-valued Carathéodory function. The definition is formally the same as in §1, except that the coefficients $f_k$ are square matrices of any fixed order $p$. The condition (1) means that the Hermitian part (or "real part") of $f(z)$ is nonnegative definite in the unit disc. In addition, we assume here that $f(z)$ is nondegenerate, in the sense that the matrix $\text{Re}(f_0)$ is invertible.

Define the $p \times p$ matrix polynomial $g_n(z)$, of formal degree $n$, and the $p \times p$ matrix function $v_n(z)$ from the identity (2), where $1$ is interpreted as the identity matrix. It can be shown that the matrix coefficients $v_{n,k}$ of $v_n(z)$ satisfy the inequality

\[
(17) \quad v_{n,k}v_{n,k}^* \leq \text{Re}(\text{tr}(f_0)) \text{Re}(f_0^{-1}),
\]

generalizing (15). Here, the star denotes the conjugate transpose, $\text{Re}$ is the Hermitian part, $\text{tr}$ stands for the trace (i.e., the sum of diagonal entries), and the inequality $a \leq b$ means that $b - a$ is nonnegative definite.

The proof of (17) is essentially the same as in §2. It is based on the Riesz-Herglotz representation (7) for nondegenerate matrix-valued Carathéodory functions and makes use of a suitable matrix extension of the Schwarz inequality [2]. Details will not be given.

References


Philips Research Laboratory, Brussels, Belgium