

PARTITIONS WITH EQUAL PRODUCTS (II)

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ABSTRACT. The following theorem is proved: Let $k \geq 3$ and r be positive integers. There exist infinitely many integers having r partitions into k parts such that the products of the integers in each partition are equal. Moreover, these partitions are mutually disjoint, i.e., no integer occurs in more than one of them.

Of some additional interest is a lemma stating that a certain class of elliptic curves has positive rank over \mathbf{Q} .

1. INTRODUCTION

In [1] we proved the following theorem:

Theorem 1. *Let k be a positive integer greater than or equal to 3. There exists an integer $N(k)$ such that every integer greater than $N(k)$ has $(k-1)$ partitions into k parts such that the products of the integers in each partition are equal. Moreover, these partitions are mutually disjoint, i.e., no integer occurs in more than one of them.*

Examples such as

$$\begin{aligned}76 + 28 + 27 &= 72 + 38 + 21 = 57 + 56 + 18 = 131 \\76 \cdot 28 \cdot 27 &= 72 \cdot 38 \cdot 21 = 57 \cdot 56 \cdot 18 = 57456\end{aligned}$$

show that in some cases more than $k-1$ partitions into k parts with equal products are possible. Here we prove a theorem which is stronger than Theorem 1 in the sense that $k-1$ is replaced by an arbitrary integer but weaker in the sense that it does not hold for all sufficiently large integers but only for infinitely many.

Theorem 2. *Let $k \geq 3$ and r be positive integers. There exist infinitely many integers having r mutually disjoint partitions into k parts with equal products.*

Note that the general case of Theorem 2 would be a trivial consequence of the case $k=3$ if we did not impose the requirement of mutual disjointness. For

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suppose that the integer N had r partitions into 3 parts with equal products. By adjoining a set of $(k - 3)$ 1's to each of these r partitions we would obtain r partitions of $N + k - 3$ into k parts with equal products.

Our proof of Theorem 2 is elementary except for the use of Mazur's theorem, [2], [3], on the torsion group of an elliptic curve in establishing Lemma 1. This lemma, as well as Theorem 3, concerning partitions into rational parts with equal products, may be of independent interest.

2. PROOF OF THEOREM 2

Lemma 1. *Let a_1, a_2 and a_3 be positive rational numbers with $a_1 > a_2 > a_3$ and*

$$(1) \quad \begin{aligned} a_1 + a_2 + a_3 &= A, \\ a_1 a_2 a_3 &= B. \end{aligned}$$

Suppose further that

$$(2) \quad a_1(a_2 - a_3)^3 \neq a_3(a_1 - a_2)^3.$$

Then the diophantine system

$$(3) \quad \begin{aligned} x_1 + x_2 + x_3 &= A, \\ x_1 x_2 x_3 &= B. \end{aligned}$$

has infinitely many positive rational solutions.

Proof. If we set $x_1 = x, x_2 = y$, we see that (3) is equivalent to the equation

$$(4) \quad xy(A - x - y) = B$$

which determines a cubic curve Γ . The critical points of $xy(A - x - y)$ are $(A/3, A/3), (0, 0), (A, 0)$ and $(0, A)$. The last three are not on Γ since $B > 0$. If $(A/3, A/3)$ satisfied (4), then we should have $B = A^3/27$ which implies, from (1), that

$$\frac{a_1 + a_2 + a_3}{3} = \sqrt[3]{a_1 a_2 a_3}.$$

Equality holds in the arithmetic-geometric mean inequality only if $a_1 = a_2 = a_3$, contrary to our hypothesis. Thus Γ is a cubic curve without a singular point, that is, a curve of genus 1. Since Γ has a rational point (a_1, a_2) , it is elliptic over the rational field \mathbf{Q} .

The real affine curve with equation (4) is displayed in Figure 1. Its three unbounded components Γ_1, Γ_2 and Γ_3 are convex with the coordinate axes and the line $x + y = A$ as asymptotes; the bounded component Γ_0 is a closed convex curve. There are six rational points on Γ_0 , namely $P_1(a_1, a_2), P_2(a_1, a_3), P_3(a_2, a_3), P_4(a_3, a_2), P_5(a_3, a_1)$ and $P_6(a_2, a_1)$ taken in clockwise order around Γ_0 . If Γ is regarded as a curve in the projective plane with homogeneous equation

$$x^2y + y^2x - Axyz + Bz^3 = 0$$

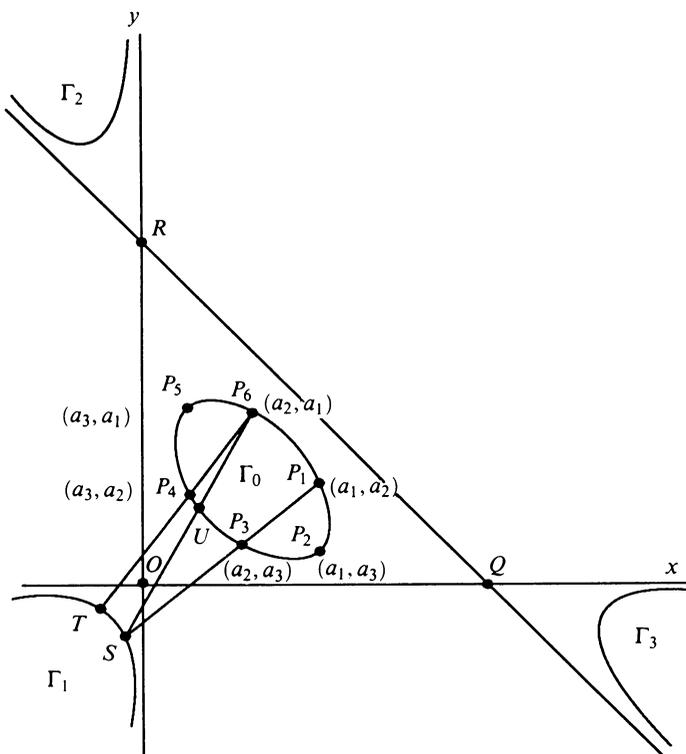


FIGURE 1.

it contains, in addition, three rational points “at infinity,” $(1, 0, 0)$, $(0, 1, 0)$ and $(1, -1, 0)$, corresponding to its asymptotes.

Lines joining certain pairs of the six rational points on Γ_0 will intersect Γ_1 , Γ_2 , and Γ_3 in additional rational points. It is routine to calculate that P_1P_3 meets Γ_1 at point S with coordinates

$$\left(\frac{-a_3(a_1 - a_2)^2}{(a_2 - a_3)(a_1 - a_3)}, \frac{-a_1(a_2 - a_3)^2}{(a_1 - a_2)(a_1 - a_3)} \right).$$

Similarly P_4P_6 meets Γ_1 at point T with coordinates

$$\left(\frac{-a_1(a_2 - a_3)^2}{(a_1 - a_2)(a_1 - a_3)}, \frac{-a_3(a_1 - a_2)^2}{(a_2 - a_3)(a_1 - a_3)} \right).$$

These two points coincide if and only if

$$a_3(a_1 - a_2)^3 = a_1(a_2 - a_3)^3$$

which is ruled out by (2). In the same manner we can find two more rational points on each of Γ_2 and Γ_3 . From Figure 1 it is clear, for example, that, since P_1P_6 and P_3P_4 are parallel to RQ , P_3P_5 and P_2P_6 must intersect Γ_2 . The condition that the two points of intersection be distinct turns out again to be (2). Likewise, P_2P_4 and P_1P_5 intersect Γ_3 in distinct points.

SP_6 meets Γ_0 at the rational point U on the shorter arc of Γ_0 strictly between P_3 and P_5 because P_3P_6 and P_5P_6 are parallel to the asymptotes of Γ_1 . (This reasoning is valid regardless of the position of the intersection of P_1S and P_6T relative to Γ_1 .) U and P_4 cannot coincide since U is on P_6S , P_4 is on P_6T and $S \neq T$. Similarly there are additional rational points on the shorter arcs of Γ_0 joining P_5 and P_1 , and P_1 and P_3 . Thus we have accumulated, including the three points at infinity, a total of eighteen rational points on Γ .

Since Γ is an elliptic curve, it may be given the structure of an algebraic variety. Mordell's theorem states that the group of rational points on Γ is a finitely generated abelian group. Mazur's theorem [2], [3] states that the torsion subgroup of this group is isomorphic to one of Z_i , $1 \leq i \leq 10$, Z_{12} , $Z_2 \times Z_2$, $Z_2 \times Z_4$, $Z_2 \times Z_6$ and $Z_2 \times Z_8$. The largest of these groups, $Z_2 \times Z_8$ has sixteen points. Since Γ has more than sixteen rational points, not all of them can be of finite order and there must be infinitely many of them.

If there were only finitely many rational points on Γ_0 , there would have to be infinitely many on at least one of the unbounded components. The lines joining each of these points with P_1 would intersect Γ_0 in infinitely many distinct rational points, a contradiction. Thus (3) has infinitely many positive rational solutions.

Corollary 1. *Under the hypotheses of Lemma 1, there exist infinitely many positive rational solutions of (3) with denominator exceeding any pre-assigned positive number, M .*

Proof. For any positive rational solution of $x_1 + x_2 + x_3 = A$ we have $0 < x_i < A$, $1 \leq i \leq 3$. It follows that the number of possible values of the x_i with denominator $\leq M$ is finite. There is essentially, i.e., apart from permutations, one solution of (3) in which a given number, say a , occurs as the value of one of the unknowns, say x_1 . For the system

$$x_2x_3 = B/a, \quad x_2 + x_3 = A - a$$

has only one solution, apart from permutations. Hence there are only finitely many solutions of (3) in positive rationals in which one or more of the unknowns have denominator $\leq M$. Therefore, by Lemma 1, there are infinitely many solutions in which all of the unknowns have denominators in excess of M .

The system

$$\begin{aligned} x + y + z &= 37, \\ xyz &= 720 \end{aligned}$$

has the solution (24, 10, 3) which violates (2). Professor Andrew Bremner has shown that this system has only finitely many rational solutions, viz., 6 permutations of (24, 10, 3), three permutations of (45, -4, -4) and three points at infinity (arising when z is eliminated). So Lemma 1 would be invalid if condition (2) were removed and not replaced by something else.

Next, we examine the consequences of (2) more closely. Let $\sigma = \{s_1, s_2, s_3\}$ be a triple of real numbers with $s_1 \geq s_2 \geq s_3$. We define

$$(5) \quad F(\sigma) = s_1(s_2 - s_3)^3 - s_3(s_1 - s_2)^3.$$

Lemma 2. *Let $\tau = \{t_1, t_2, t_3, t_4\}$ be a set of four distinct positive numbers. Define $\tau_i = \tau - \{t_i\}$, $1 \leq i \leq 4$. At most two of the four quantities $F(\tau_i)$, $1 \leq i \leq 4$, can vanish.*

Proof. We may assume that $t_1 > t_2 > t_3 > t_4 > 0$. Clearly $F(\tau_3) > F(\tau_4)$ and $F(\tau_3) > F(\tau_2)$ so if $F(\tau_3) \leq 0$, the lemma is evident. If $F(\tau_3) > 0$ and $F(\tau_1) = F(\tau_2) = 0$ we have

$$\frac{(t_3 - t_4)^3}{t_4} = \frac{(t_1 - t_3)^3}{t_1} = \frac{(t_2 - t_3)^3}{t_2},$$

whence

$$t_1(t_2 - t_3)^3 = t_2(t_1 - t_3)^3,$$

so that

$$F(\tau_4) = t_2(t_1 - t_3)^3 - t_3(t_1 - t_2)^3 > 0.$$

Lemma 3. *Let a_1, a_2, a_3, a_4 be distinct positive rational numbers such that*

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= A, \\ a_1 a_2 a_3 a_4 &= B. \end{aligned}$$

Let M be a positive number. There exist distinct positive rational numbers b_1, b_2, b_3, b_4 , all with denominator greater than M , such that

$$\begin{aligned} b_1 + b_2 + b_3 + b_4 &= A, \\ b_1 b_2 b_3 b_4 &= B. \end{aligned}$$

Proof. By Lemma 2 we may assume that $a_1 > a_2 > a_3 > 0$ and

$$a_1(a_2 - a_3)^3 \neq a_3(a_1 - a_2)^3.$$

Corollary 1 tells us that the system

$$\begin{aligned} x_1 + x_2 + x_3 &= A - a_4, \\ x_1 x_2 x_3 &= B/a_4 \end{aligned}$$

has infinitely many positive rational solutions with denominator greater than M . The number of solutions with two of the unknowns equal is finite as is the number with one of them equal to a_4 . Hence we can find a solution (d_1, d_2, d_3) with the d_i positive rationals distinct from each other and from a_4 and having denominators greater than M . Applying Lemma 2, we conclude that F must fail to vanish on some 3-subset of $\{d_1, d_2, d_3, a_4\}$ containing a_4 . Assume then, that $F(\{d_1, d_2, a_4\}) \neq 0$. Note that the system

$$(6) \quad \begin{aligned} x_1 + x_2 + x_3 &= A - d_3, \\ x_1 x_2 x_3 &= B/d_3 \end{aligned}$$

has solution (d_1, d_2, a_4) . Reasoning as before from Corollary 1, we see that the system (6) has a solution (e_1, e_2, e_3) in positive rationals which are distinct from each other and from d_3 and which have denominators greater than M . Setting $b_1 = e_1, b_2 = e_2, b_3 = e_3, b_4 = d_3$ completes the proof of Lemma 3.

Lemma 4. *Let $k \geq 4$. Let a_1, \dots, a_k be distinct positive rationals such that*

$$a_1 + \dots + a_k = A, \quad a_1 \cdots a_k = B.$$

Let M be a positive number. There exist distinct positive rationals b_1, \dots, b_k with denominators greater than M such that

$$b_1 + \dots + b_k = A, \quad b_1 \cdots b_k = B.$$

Proof. We proceed by induction on k . The case $k = 4$ is the previous lemma. Assume the lemma for $k = n$ and let a_1, a_2, \dots, a_{n+1} be $n + 1$ distinct positive rationals such that

$$a_1 + \dots + a_{n+1} = A, \quad a_1 \cdots a_{n+1} = B.$$

Let M' be the denominator of a_{n+1} and let $M^* = \max(M, M')$. By the inductive hypothesis there exist distinct positive rationals d_1, \dots, d_n with denominator $> M^*$ such that

$$d_1 + \dots + d_n = A - a_{n+1}, \quad d_1 \cdots d_n = B/a_{n+1}.$$

These rationals are also distinct from a_{n+1} , having a higher denominator. Note that

$$d_1 + \dots + d_n + a_{n+1} = A, \quad d_1 d_2 \cdots d_n a_{n+1} = B.$$

Now consider the system

$$(7) \quad \begin{aligned} x_1 + \dots + x_{n+1} &= A - d_1, \\ x_2 \cdots x_{n+1} &= B/d_1. \end{aligned}$$

(7) has the solution $x_i = d_i, 2 \leq i \leq n, x_{n+1} = a_{n+1}$ in distinct positive rationals. Let M'' be the denominator of d_1 . Applying the inductive hypothesis again, we see that (7) has a solution $x_i = b_i, 2 \leq i \leq n + 1$ in distinct positive rationals with denominator $> M'' > M$. These rationals are distinct from d_1 , having a higher denominator. Setting $b_1 = d_1$ establishes the case $k = n + 1$ and completes the proof of the lemma.

Theorem 3. *Let $k \geq 3$. Let a_1, a_2, \dots, a_k be distinct positive rationals such that*

$$a_1 + a_2 + \dots + a_k = A, \quad a_1 a_2 \cdots a_k = B.$$

If $k = 3$, assume in addition, that

$$F(\{a_1, a_2, a_3\}) \neq 0;$$

then the system

$$x_1 + x_2 + \dots + x_k = A, \quad x_1 x_2 \cdots x_k = B$$

has an infinite family of solutions in distinct positive rationals such that no rational number occurs in more than one solution of the family.

Proof. Repeatedly apply Corollary 1 if $k = 3$ and Lemma 4 if $k \geq 4$. At the n th step let $M = M_n =$ maximum denominator of all previously obtained solutions. In the case $k = 3$ we also use the fact that the system

$$x_1 + x_2 + x_3 = A, \quad x_1 x_2 x_3 = B$$

has only finitely many solutions in which the unknowns are not distinct.

Lemma 5. *Let k and ℓ be positive integers with $k \geq 3$. There exists an integer $N(k, \ell)$ having ℓ mutually disjoint partitions into k parts, all with the same product.*

Proof. Let A and B be two rationals to which Theorem 3 applies. Find ℓ positive rational solutions of the system

$$(8) \quad \begin{aligned} x_1 + \cdots + x_k &= A, \\ x_1 \cdots x_k &= B \end{aligned}$$

so that no rational number occurs in more than one of them. Let D be the least common multiple of all the denominators in all these solutions. Let $N(k, \ell) = AD$. The system

$$\begin{aligned} y_1 + \cdots + y_k &= AD, \\ y_1 \cdots y_k &= BD^k \end{aligned}$$

has ℓ mutually disjoint integral solutions obtained by putting $y_i = Dx_i$, $1 \leq i \leq k$ in each of the ℓ solutions of (8).

Proof of Theorem 2. The set

$$N_{k,r} = \{N(k, \ell) : \ell \geq r\}$$

contains infinitely many integers since any particular integer has only a finite number of partitions with k parts. Any integer in $N_{k,r}$ satisfies the conclusion of Theorem 2.

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