NOTES OF THE INVERSION OF INTEGRALS I

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Abstract. If \( W \) is a Picard bundle on the Jacobian \( J \) of a curve \( C \), we have the problem of describing \( W \) globally. The theta divisor \( \theta \) is ample on \( J \). Thus it is possible to write \( n^*W \) as the sheaf associated to a graded \( M \) over the well-known ring \( \oplus_{m \geq 0} \Gamma(J, \mathcal{O}_J(m \theta)) \). In this paper we compute the degree of generators and relations for such a module \( M \).

There are naturally occurring locally free sheaves called Picard bundles on the Jacobian \( J \) of a smooth complete curve \( C \) of positive genus \( g \) over \( k = \overline{k} \). These bundles describe the global variation of the sections of invertible sheaves on \( C \) with pleasant degree.

The inversion problem is to give a description of the Picard bundles globally on \( J \). As such analytic description is lacking, we must content ourselves with two algebraic solutions of this problem.

The first solution requires us to know the image of some points of \( C \) in the Jacobian. This approach uses a method due to R. C. Gunning. The second solution determines the pull-back of the Picard bundle by a multiplication in \( J \) in terms of a module over the graded ring of theta sections. Here one uses a form of a theorem of D. Mumford on the equations defining abelian varieties projectively.

1. THE FIRST METHOD

Let \( \mathcal{P} \) be a Poincaré sheaf on \( J \times J \). Let \( L_n \) be an invertible sheaf on \( J \) of the form \( \mathcal{O}_J(n \theta) \) where the divisor \( \theta \) gives the usual principal polarization of \( J \). If \( n > 0 \) then \( \pi_2^* L_n \otimes \mathcal{P} \) is a family of ample invertible sheaves on the second factor. It follows from Mumford’s vanishing theorem that

\[
R^i \pi_1^*(\pi_2^* L_n \otimes \mathcal{P})
\]

is zero if \( i > 0 \) and \( \mathcal{V}_n = \pi_1^*(\pi_2^* L_n \otimes \mathcal{P}) \) is a locally free sheaf of rank \( n^g \).

Let \( C \to J \) be a universal abelian integral. Let \( Q_n = \pi_2^* L_n \otimes \mathcal{P}|_{J \times C} \). Then \( Q_n \) is a family of invertible sheaves on \( C \) of degree \( n \cdot g \) as \( [C : \theta] = g \). If

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Let $D$ be an effective divisor on $C$ of degree $d$. Then $Q_n(-J \times D)$ is a family of invertible sheaves on $C$ of degree $n \cdot g - d$. If $d > n \cdot g$, then $\pi_J(Q_n(-J \times D))$ is zero and the Picard sheaf $\mathcal{U}_n(D) \equiv R^1\pi_J(Q_n(-J \times D))$ is locally free of rank $d - n \cdot g + g - 1 = d - (n - 1)g - 1$. Consider the exact sequence

$$0 \to Q_n(-J \times D) \to Q_n \to Q_n|_{J \times D} \to 0.$$  
This yields the well-known exact sequence of

**Lemma 2.** We have an exact sequence

$$0 \to \mathcal{W}_{ng} \xrightarrow{\epsilon} \pi_J(Q_n|_{J \times D}) \to \mathcal{U}_n(D) \to 0$$

where $\epsilon$ is just evaluation.

The composition $\beta_D: \mathcal{U}_n \xrightarrow{\alpha} \mathcal{W}_{ng} \xrightarrow{\epsilon} \pi_J(Q_n|_{J \times D}) = \pi_J(\pi_2^n \mathcal{L} \otimes \mathcal{P}|_{J \times D})$ is simply restriction and is determined only by how $D$ sits as a closed subscheme of $J$. The combination of the above facts give the first solution of the inversion problem.

**Theorem 3.** $\mathcal{W}_{ng} = \text{Image}(\beta_D)$ and $\mathcal{U}_n(D) = \text{Cokernel}(\beta_D)$.

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**2. Normal presentation**

Let $\mathcal{L}$ be a very ample sheaf on a projective variety $X$. A coherent sheaf $\mathcal{F}$ on $X$ is said to be normally presented if we have an exact sequence

$$R \otimes_k \mathcal{L} \otimes^{-1} \xrightarrow{\alpha} G \otimes_k \mathcal{O}_X \to \mathcal{F} \to 0$$

for some vector spaces $R$ and $G$. Furthermore $\mathcal{F}$ is said to be strongly presented if the homomorphism $G \to \Gamma(X, \mathcal{F})$ is surjective.

**Lemma 4.** A strongly presented coherent sheaf $\mathcal{F}$ is determined by $\Gamma(X, \mathcal{F})$ and the kernel of the multiplication

$$\Gamma(X, \mathcal{F}) \otimes \Gamma(X, \mathcal{L}) \to \Gamma(X, \mathcal{F} \otimes \mathcal{L}).$$

**Proof.** First of all we may assume that $\beta: G \to \Gamma(X, \mathcal{F})$ is an isomorphism by factoring $G \otimes_k \mathcal{O}_X \to \mathcal{F}$ through $\overline{G} \otimes_k \mathcal{O}_X$, where $\overline{G}$ is the image of $\beta$. Then $R \to G \otimes \Gamma(X, \mathcal{L})$ has image in the kernel of multiplication. Hence the kernel contains enough relations to define $\mathcal{F}$ as a quotient sheaf of $G \otimes_k \mathcal{O}_X$. □

We will need a lemma to prove that some sheaves are strongly presented.
Lemma 5. Given an exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

of coherent sheaves on $X$, assume that

(a) $\mathcal{F}_2$ is strongly normally presented, and

(b) $\mathcal{F}_1$ is generated by its sections and $H^1(X, \mathcal{F}_1)$ is zero. Then $\mathcal{F}_3$ is strongly normally presented.

Proof. By (b) we have an exact sequence, $0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to 0$ and a surjection $\Gamma(X, \mathcal{F}_1) \otimes_k \mathcal{F}_X \to \mathcal{F}_1$. By (a) we have an exact sequence

$$R \otimes_k \mathcal{L}^{-1} \to \Gamma(X, \mathcal{F}_2) \otimes \mathcal{O}_X \to \mathcal{F}_2 \to 0$$

(using the proof of Lemma 4). Therefore we have an exact sequence

$$R \otimes_k \mathcal{L}^{-1} \oplus \Gamma(X, \mathcal{F}_1) \otimes_k \mathcal{O}_X \to \Gamma(X, \mathcal{F}_2) \otimes \mathcal{O}_X \to \mathcal{F}_3 \to 0$$

which we can factor as

$$R \otimes_k \mathcal{L}^{-1} \to \Gamma(X, \mathcal{F}_3) \otimes \mathcal{O}_X \to \mathcal{F}_3 \to 0.$$  

Remark. If we just assume that $\mathcal{F}_1$ is generated by its sections, then we can conclude that $\mathcal{F}_3$ is normally presented.

3. Abelian varieties

Let $X$ be an abelian variety with ample invertible sheaf $\mathcal{L}$. An invertible sheaf $\mathcal{M}$ on $X$ is said to be of type $n$ if it is algebraically equivalent to $\mathcal{L}^\otimes n$ for some integer $n$. If the type is $\mathcal{M} \geq 1$ then $\mathcal{M}$ is ample and if it is $\geq 2$ then $\mathcal{M}$ is generated by its sections and if it is $\geq 3$ then $\mathcal{M}$ is very ample.

We have a basic result.

Theorem 6. If $\mathcal{N}$ and $\mathcal{M}$ are two invertible sheaves on the abelian variety $X$ such that $\text{type}(\mathcal{N}) \geq 3$ and $\text{type}(\mathcal{M}) \geq 4$, then $\mathcal{N}$ is strongly normally generated for $\mathcal{M}$.

Proof. We first need to write enough relations between the sections of $\mathcal{N}$ and $\mathcal{M}$. Let $Q_n$ be an invertible sheaf of type 2. We may write $\mathcal{N} = \mathcal{L}_n \otimes Q_n$ and $\mathcal{M} = \mathcal{L}_n \otimes Q_n$, where $\text{type}(\mathcal{L}_n) \geq 1$ and $\text{type}(\mathcal{L}_n) \geq 2$. Let $r \in \Gamma(X, \mathcal{L}_n)$, $s \in \Gamma(X, \mathcal{L}_n)$ and $q_1$ and $q_2 \in \Gamma(X, \mathcal{O}_n)$. Let $\langle , \rangle$ denote the product of two sections. Evidently

$$a(r, s, q_1, q_2) = \langle r, q_1 \rangle \otimes \langle s, q_w \rangle - \langle r, q_2 \rangle \otimes \langle s, q_1 \rangle$$

is contained in the kernel of the multiplication

$$\Gamma(X, \mathcal{N}) \otimes \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{N} \otimes \mathcal{M}).$$

Let $A$ be the span of all possible such relations $a(r, s, q_1, q_2)$ for all possible $\alpha$. 

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Let $N = \Gamma(X, \mathcal{N}) \otimes_k B/AB$ where $B$ is the graded ring $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{M}^\otimes n)$. We have a canonical surjection $\pi: \tilde{N} \to \mathcal{N}$ where $\tilde{N}$ is the $\mathcal{O}_X$-module associated to the $A$-module $N$. The theorem will be proven if we can show that $\pi$ is an isomorphism.

To do the above we must show that (1) for any point $x$ of $X$ the vector space $\tilde{N}(x)$ is one-dimensional.

We may assume that all sheaves on $X$ have been given compatible trivialization at $x$ and let $e(\sigma) = \sigma(x)$ be evaluation. Then $\tilde{N}(x) = \Gamma(X, \mathcal{N})/(1 \otimes e)A$ by definition. Thus we need to show that $(1 \otimes e)A$ has codimension one in $\Gamma(X, \mathcal{N})$. As the whole theorem is invariant under translation we may assume that $x$ is the identity $0$ of $X$.

Take $\lambda: \Gamma(X, \mathcal{N}) \to k$ a linear functional such that $\lambda((1 \otimes e)A) = 0$. Now $(1 \otimes e)a(r,s,q_1,q_2) = ((1 \otimes s(0))((r,q)q_2(0) - (r,q)q_1(0)).$ As $S_n$ is generated by its sections we may assume that $s(0) \neq 0$. Therefore $\lambda((r,q)q_2(0)) = \lambda((r,q)q_1(0))$ is symmetric in $q_1$ and $q_2$ and vanishes if $q_1(0)$ or $q_2(0)$ equals zero. Write $\lambda((r,q)q_2(0)) = \mu(r)q_1(0)q_2(0)$ and note that $\mu$ is well defined because $Q_n$ is generated by its sections. We intend to show that (2) $\mu_n(r) = \text{constant} \cdot r(0)$.

We will show that (2) follows from a global variational argument with $\alpha$. Let $\mathcal{P}$ be a Poincaré sheaf on $X \otimes \hat{X}$ where $\hat{X}$ is the dual abelian variety. Let $\mathcal{R}$ be one possible choice of $\mathcal{K}$. Then all possible choices are the restriction of $\pi_x^* \mathcal{R} \otimes \mathcal{P}$ to the fibers of $\pi_x^* \hat{X}$. Globally $\mu_n$ is the value of a $\mathcal{O}_x^*$-homomorphism $\mu: \mathcal{W} \equiv \pi_x^* (\pi_x^* \mathcal{R} \otimes \mathcal{P}) \to \mathcal{O}_x^*$. By [1,4] $H^{\dim \hat{X}}(\hat{X}, \mathcal{W})$ is onedimensional. Hence by duality $\text{Hom}_{\hat{X}}(\mathcal{W}, \mathcal{O}_{\hat{X}})$ is onedimensional but evaluation at 0 is one such homomorphism. Hence $\mu$ is a multiple of evaluation. Therefore (2) is true. $\square$

When $\mathcal{N} = \mathcal{M}$ the theorem follows from D. Mumford’s theorem [3, 4] that $\bigoplus_{k \geq 0} \Gamma(X, \mathcal{M}^\otimes k)$ is almost normally presented as a ring. The proof of the theorem is close to Mumford’s reasoning but the technicalities are easier.

4. The second method

An invertible sheaf $\mathcal{L}$ on the Jacobian $J$ has type $n$ if $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_J(n\theta)$ where $\theta$ is in the class of the principal polarization. Thus type$(\mathcal{L}_n) = n$ where $\mathcal{L}_n$ is the sheaf of $\S 1$, the notation of which we will be using.

Let $\mathcal{M}(\mathcal{R})$ be invertible sheaves on $J$ of type $m(r)$. One might hope to prove that $\mathcal{H}_n(D) \otimes \mathcal{M}$ is normally presented for $\mathcal{R}$ for reasonable bounds on $m$, $n$ and $r$. If one tries to use Lemma 2 and Lemma 5, the problem is that we would need $\mathcal{H}_n \otimes \mathcal{M}$ to be generated by its sections (but I do not now when this is true). This emphasis is circumvented by applying the isogeny
$nL_j : J \to J$ given by multiplication by $n$. This resolves the problem. The result is

**Theorem 7.** (a) $((n_1j)^*\mathcal{U}_n(D)) \otimes \mathcal{M}$ is normally presented for $\mathcal{R}$ if $m \geq n + 2 \geq 4$ and $r \geq 4$.

(b) It is strongly normally presented for $\mathcal{R}$ if $m \geq n + 2 \geq 4$, $r \geq 4$ and, if $n \geq 3$, then $g \geq 2$ and $m/(m,n)$ prime to char($k$).

**Proof.** First of all we may assume that the effective divisor $D$ consists of distinct point $e_1, \ldots, e_d$. This follows because the isomorphism class of $\mathcal{U}_n(D)$ only depends on that of $\mathcal{L}_n | C \equiv \mathcal{H}$ but we may vary $\mathcal{L}_n$ and $D$ so that $D$ is reduced while not changing $\mathcal{H}$.

Then from Lemma 2 we have an exact sequence

$$0 \to \mathcal{W}_n \to \bigoplus_{1 \leq i \leq d} \mathcal{S}_i \to \mathcal{U}_n(D) \to 0$$

where $\mathcal{S}_i = \pi_j(Q_n|_{J \times e_i})$. Now type($\mathcal{S}_i$) = 0 because

$$Q_n|_{J \times e_i} = (\pi_j^*\mathcal{L}_n \otimes \mathcal{P})|_{J \times e_i} = \mathcal{L}_n(e_i) \otimes _k \mathcal{P}|_{J \times e_i},$$

which is algebraically equivalent to $\mathcal{E}_j$.

Next we pull this sequence back and get

1. $0 \to (n_1j)^*\mathcal{W}_n \to \bigoplus_{1 \leq i \leq d} \mathcal{S}_i \to (n \times 1_j)^*\mathcal{U}_n(D) \to 0$, where type($\mathcal{S}_i$) = 0 and $\mathcal{S}_i = (n_1j)^*\mathcal{S}_i$. Now $\bigoplus_{1 \leq i \leq d} \mathcal{S}_i \otimes \mathcal{M}$ is strongly normally presented for $\mathcal{R}$ if $m \geq 3$ and $r \geq 4$ by Theorem 6. Thus point (a) will follow from $\bigoplus \otimes \mathcal{M}$ and the remark after Lemma 5 if we can prove that

2. $(n_1j)^*\mathcal{W}_n \otimes \mathcal{M}$ is generated by its sections if $m \geq n + 2$. Also by Lemma 5 the point (b) will follow if we prove

3. $H^1(J, (n_1j)^*\mathcal{W}_n \otimes \mathcal{M}) = 0$ if $m \geq n + 2 \geq 4$ and if $n \geq 3$ then $\frac{m}{n/(n - 2))^{g-1}}$ and $m/(m,n)$ prime to char($k$).

To prove (2) we will use the surjection $\alpha : \mathcal{V}_n \to \mathcal{W}_n$ of Proposition 1 as $n \geq 2$. Thus (2) will follow if we prove

4. $(n_1j)^*\mathcal{U}_n \otimes \mathcal{M}$ is generated by its sections if $m \geq n + 2$.

By [1] $(n_1j)^*\mathcal{V}_n \approx \Gamma(J, \mathcal{L}_n) \otimes _k \mathcal{L}_n^{\otimes -1}$. Hence we just need $(\mathcal{L}_n^{\otimes -1} \otimes \mathcal{R})$ to be generated by its sections; e.g. its type $\geq 2$. As the type is $m - n$, (4) is true.

To prove (3) we have to modify the argument of [1] due to the presence of $(n_1j)^*$. We will first give some isomorphisms which follow in the same way as [1] from the vanishing of higher direct images and the Leray spectral sequence.

$$H^i(J, (n_1j)^*\mathcal{W}_n \otimes \mathcal{M} \simeq H^i(J \times C, \pi_j^*\mathcal{L}_n \otimes \mathcal{P} \otimes \pi_j^*\mathcal{M}|_{J \times C})$$
as
\[(n_1^n)^\ast \mathcal{W}_g = \pi_j.((n_1^n \times 1_C)^\ast (\pi_\ast \mathcal{L}_n \otimes \mathcal{P})|_{J \times C}) \]
\[= \pi_j.((\pi_\ast \mathcal{L}_n \otimes \mathcal{P}^{\otimes n})|_{J \times C}). \]

\[H^i(J \times C, \pi_\ast \mathcal{L}_n \otimes \mathcal{P} \otimes \pi^\ast \mathcal{M}|_{J \times C}) = H^i(C, \mathcal{L}_n \otimes (n_1^n)^\ast (\pi_2. (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M})))|_C \]
as \[\pi_\ast \mathcal{L}_n|_C \otimes (\mathcal{P}^{\otimes n} \otimes \pi^\ast_1 \mathcal{M})|_{J \times C}) \simeq \pi_2.((\pi_\ast \mathcal{L}_n \otimes (1_j \times n_1^n)^\ast (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M})))|_C \simeq \mathcal{L}_n \otimes n_1^n)^\ast (\pi_2. (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M})))|_C \]

Thus
\[H^i(J(n_1^n)^\ast \mathcal{W}_g \otimes \mathcal{M}) \simeq H^i(C, \mathcal{L}_n \otimes (n_1^n)^\ast (\pi_2. (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M})))|_C \]

Now we need to determine when the cohomology of this sheaf on \(C\) vanishes. Let \(a = m/(m,n)\). Write \(m = ab\) and \(n = cb\). As \(a\) is prime to the characteristic, the curve \(C_a = (a_1^n)^{-1} C\) is an unramified Galois covering of \(C\). Hence for any quasicoherent sheaf \(\mathcal{F}\) on \(C\) we have an injection \(H^i(C \mathcal{F}) \hookrightarrow H^i(C_a, (a_1^n)^\ast \mathcal{F})\). Thus we want to study

\[H^1(C_a, (a_1^n)^\ast (\mathcal{L}_n \otimes (n_1^n)^\ast (\pi_2. (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M})))|_{C_a}) \]

Here
\[(a_1^n)^\ast (\mathcal{L}_n \otimes (n_1^n)^\ast (\pi_2. (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M}))) \]
\[\simeq (a_1^n)^\ast (\mathcal{L}_n \otimes (c_1^n)^\ast (m_1^n)^\ast (\pi_2. (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M}))) \]
\[\simeq (a_1^n)^\ast (\mathcal{L}_n \otimes (c_1^n)^\ast \mathcal{M}^{\otimes -1} \otimes \Gamma(J, \mathcal{M}) \]
as \[(m_1^n)^\ast (\pi_2. (\mathcal{P} \otimes \pi^\ast_1 \mathcal{M}))) \simeq \mathcal{M}^{\otimes -1} \otimes \Gamma(J, \mathcal{M}) \]

Thus the first cohomology groups vanish if
\[\deg((a_1^n)^\ast (\mathcal{L}_n \otimes (c_1^n)^\ast \mathcal{M}^{\otimes -1})|_{C_a}) > 2(\text{genus}(C_a) - 1) \]
\[= 2 \deg(a_1^n)(g - 1) \]
\[= 2a^{2g}(g - 1) \]

but the degree of the sheaf is \(a^{2g}ng - c^{2g}mg\). Finally for the vanishing we need the inequality \(a^{2g}gn - c^{2g}gm > 2a^{2g}(g - 1)\) or rather \(2m^{2g-1}/gn + m^{2g-1} > n^{2g-1} + (2/n)m^{2g-1}\). For \(n \geq 3\), this is true if \(m > (n/(n - 2))^{1/2g-1}\) or, if \(n = 2\) if \(m > 2^{1/2g-1}\). The simplest case of part (b) is \(n = 3\) and \(m = 5\) if \(g \geq 2\) or \(n = 2\) and \(m = 4\). \(\square\)

The theorem tells us when \(((n_1^n)^\ast \mathcal{U}_n(D)) \otimes \mathcal{M}\) is determined by the multiplication
\[\beta : \Gamma(J, ((n_1^n)^\ast \mathcal{U}_n(D)) \otimes \mathcal{M}) \otimes \Gamma(J, \mathcal{R}) \hookrightarrow \Gamma(J, ((n_1^n)^\ast \mathcal{U}_n(D)) \otimes \mathcal{M} \otimes \mathcal{R}).\]

Thus we want to know more about this group. I will only give the dimension here.
Theorem 8. (a) $H^i(J, ((n_1 j)^* \mathcal{U}_n(D)) \otimes \mathcal{M}) = 0$ if $i > 0$ and $m \geq 1$.

(b) $\dim \Gamma(J, ((n_1 j)^* \mathcal{U}_n(D)) \otimes \mathcal{M}) = ((d - gn + g - 1)m^g + gn^2 m^{g-1})$ if $m \geq 1$.

Proof. Consider the long exact sequence of cohomology of the sequence (1) tensored with $\mathcal{M}$. If $m \geq 1$ then the higher cohomology groups of the sheaf $\mathcal{T}_j \otimes \mathcal{M}$ vanish and by the isomorphism $\otimes$ of the last proof as

$$H^i(J, (n_1 j)^* \mathcal{U}_n \otimes \mathcal{M}) = 0 \quad \text{if} \quad i \geq 1 = \dim C.$$

Thus (a) is true by the long exact sequence.

By (a) the dimension is the Euler characteristic $\chi((n_1 j)^* \mathcal{U}_n(D) \otimes \mathcal{M})$, which I intend to compute using the Hirzebruch-Riemann-Roch theorem. We need to know $\mathrm{ch}((n_1 j)^* \mathcal{U}_n(D) \otimes \mathcal{M})$ because its number of codimension $g$ cycles (= points) is the Euler characteristic as the Todd class of $J$ is 1.

By Mattuck's result $c_i(\mathcal{U}_n(D)) = \sum_{i \geq 0} w_i t^i$ is algebraic equivalent where $w_i$ is the image of $C^{(g-1)}$ in $J$. Thus by Poincaré relation, $c_i(\mathcal{U}_n(D)) = \exp(\theta t)$ in numerical equivalence. Now if $c_i(\mathcal{U}_n(D)) = \prod_{1 \leq i \leq n} 1 + k_i t$, we get $+ t \theta = \log(\exp(\theta t)) = + \sum_i \log(1 + w_i t) = \sum_{p \geq 1} \sum_i (-1)^p w_i^{p \theta} / p$ but $\mathrm{ch}((n_1 j)^* \mathcal{U}_n(D)) = \sum_i \exp(w_i) = \sum_p \sum_i w_i^{p \theta} / p$. Comparing coefficients we find

$$\mathrm{ch}_i(\mathcal{U}_n(D)) = \mathrm{rank}(\mathcal{U}_n(D)) + t c_1(\mathcal{U}_n(D)) = \mathrm{rank} + t \theta,$$

where $\mathrm{rank} = -g \cdot n + d + g = 1$. Hence $\mathrm{ch}((n_1 j)^* \mathcal{U}_n(D)) = \mathrm{rank} + n^2 \theta$. Thus

$$\mathrm{ch}((n_1 j)^* \mathcal{U}_n(D) \otimes \mathcal{M}) = \mathrm{ch}(\mathcal{M} \otimes \mathcal{U}_n(D)) \cdot \mathrm{ch}(\mathcal{M}) = \mathrm{rank} + n^2 \theta \exp(m \theta).$$

Therefore $\chi((n_1 j)^* \mathcal{U}_n(D) \otimes \mathcal{M}) = \mathrm{rank} m^g + n^2 m^{g-1} g$ and the result follows. □

A last remark is

Theorem 9. In the range of Theorem 7(b) then the multiplication $\beta$ is surjective.

Proof. The conditions of Theorem 7(b) are true when $\mathcal{M} = \mathcal{M} \otimes \mathcal{R}$. The proof shows that the homomorphism

$$\Gamma(J, \mathcal{T}_j \otimes \mathcal{M}) \rightarrow \Gamma(J, (m_1 j)^* \mathcal{U}_n(D) \otimes \mathcal{M})$$

is surjective for $\mathcal{M} = \mathcal{M}$ and $\mathcal{M} \otimes \mathcal{R}$. This theorem results because the multiplication

$$\Gamma(J, \mathcal{T}_j \otimes \mathcal{M}) \otimes \Gamma(J, \mathcal{R}) \rightarrow \Gamma(J, \mathcal{T}_j \otimes \mathcal{M} \otimes \mathcal{R})$$

is surjective by Mumford's result in [4]. □

References


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