O(2) x O(2)-INARIANT HYPERSURFACES WITH CONSTANT NEGATIVE SCALAR CURVATURE IN E^4

TAKASHI OKAYASU

Abstract. We use the method of equivariant differential geometry to prove the existence of a complete hypersurface with constant negative scalar curvature in E^n (n ≥ 4). This is the first example of a complete hypersurface with constant negative scalar curvature in E^n (n ≥ 4).

1. Introduction

Let (G, En+1) be an isometric transformation group of the (n + 1)-dimensional Euclidean space En+1 with codimension two principal orbit type. Such orthogonal transformation groups are classified in [2]. It is natural and interesting to study hypersurfaces in En+1 with constant scalar curvature, which are invariant with respect to one of such orthogonal transformation group. In [1,4], O(n)-invariant hypersurfaces with constant scalar curvature in real space forms are classified completely. In particular, it is proved that all O(n)-invariant complete hypersurfaces with constant scalar curvature in E^{n+1} have nonnegative scalar curvature. In this paper, we consider O(2) x O(2)-invariant hypersurfaces in E^4. We show that there is an O(2) x O(2)-invariant complete hypersurface M^3 of constant negative scalar curvature in E^4. Making product M^3 with E^{n-3}, we obtain a complete hypersurface with constant negative scalar curvature in E^{n+1}. Note that these spaces are the first examples of complete hypersurfaces of constant negative scalar curvature in the Euclidean spaces.

2. O(2) x O(2)-INARIANT HYPERSURFACES

We consider (O(2) x O(2), E^4), where O(2) x O(2) acts orthogonally on E^4 via the representation ρ_2 ⊕ ρ_2'. We need the following fact [3].

(i) The orbit space of O(2) x O(2)-action on E^4 can be parametrized by Q: = {(x, y); x ≥ 0, y ≥ 0} with the orbital distance metric ds^2 = dx^2 + dy^2.
(ii) Interior points of $Q$ correspond to principal orbits which are of the type $S^1 \times S^1$. More precisely, the inverse image of $(x, y)$ in the interior of $Q$ is a product $S^1(x) \times S^1(y)$.

We consider a smooth curve $\gamma(s) = (x(s), y(s))$ in the interior of $Q$, which is parametrized by the arc length (i.e. $(dx/ds)^2 + (dy/ds)^2 = 1$). We denote the $O(2) \times O(2)$-invariant hypersurface in $E^4$ generated by $\gamma$ as $M_\gamma$. The induced metric $g$ on $M_\gamma$ is $g = x(s)^2 g_1 + y(s)^2 g_2 + ds^2$, where $g_1, g_2$ are the canonical metrics on $S^1(1)$, respectively.

**Proposition 1.** $M_\gamma$ is of constant scalar curvature $a$ if and only if $\gamma$ satisfies the following differential equation.

\[
(x'y'' - y'x'') \left( -\frac{x'}{y} + \frac{y'}{x} \right) - \frac{x'y'}{xy} = \frac{a}{2},
\]

where $x'$, $y'$, $x''$, $y''$ are derivatives with respect to $s$.

**Proof.** Since the principal curvatures of $M_\gamma$ in $E^4$ are $x'y'' - y'x''$, $-x'/y$ and $y'/x$, we get the conclusion.

Using $(dx/ds)^2 + (dy/ds)^2 = 1$ and $dy/ds = (dy/dx)(dx/ds)$, we write (1) as

\[
\frac{d^2y}{dx^2} = \left\{ 1 + \left( \frac{dy}{dx} \right)^2 \right\} \left\{ \frac{dy}{dx} + \frac{axy}{2} \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) \right\} \left( y \frac{dy}{dx} - x \right)^{-1}.
\]

From now on we assume that $a \leq 0$. We denote by $f(x, a)$ the solution of (2) satisfying the initial condition $y(1) = 1$, $y'(1) = -1$. Since (1) and (2) are symmetric with respect to $x$ and $y$, we only consider $x \geq 1$ part of $y = f(x, a)$. From the elementary ODE theory, we see that $y = f(x, a)$ exists at least on the interval $[1, x']$, where $x'$ is the least point which satisfies $f(x', a) = 0$ or $(df/dx)(x, a) = 0$.

**Lemma 1.** \textit{If $f(x, a) > 0$ and $(\partial f/\partial x)(x, a) < 0$, then $g(x', a) = 0$.}

**Proof.** From $a \leq 0$ and (2) we easily get the conclusion.

**Lemma 2.** \textit{If there is $x_0 > 1$ such that $f(x_0, a) = 0$ and $f(x, a) > 0$, $(\partial f/\partial x)(x, a) < 0$ for all $x \in [1, x_0)$, then $(\partial f/\partial x)(x_0, a) < 0$ (i.e. $y = f(x, a)$ intersects with the $x$-axis transversally).}

**Proof.** If $(\partial f/\partial x)(x_0, a) = 0$, then $y = 0$ and $y = f(x, a)$ are two distinct solutions of (2) with the same initial condition $y(x_0) = 0$, $(dy/dx)(x_0) = 0$. This is a contradiction.

By Lemmas 1 and 2, we can classify the solution $y = f(x, a)$ ($x \geq 1$).

**Lemma 3.** \textit{For any $a \leq 0$, $y = f(x, a)$ ($x \geq 1$) is one of the following three types (see Figure 1).}
(a) There is $x_0 > 1$ such that $f(x_0, a) = 0$ and $f(x, a) \geq 0$, $(\partial f/\partial x)(x, a) < 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ for all $x \in [1, x_0]$. In this case $y = f(x, a)$ intersects with the $x$-axis transversally.

(b) There is $x_0 > 1$ such that $(\partial f/\partial x)(x_0, a) = 0$ and $f(x, a) \geq f(x_0, a) > 0$, $(\partial f/\partial x)(x, a) \leq 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ for all $x \in [1, x_0]$.

(c) $y = f(x, a)$ is defined globally on $[1, \infty)$ and $f(x, a) > 0$, $(\partial f/\partial x)(x, a) < 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ for all $x \in [1, \infty)$. If $a < 0$, the $x$-axis is the asymptotic line of $y = f(x, a)$. If $a = 0$, $y = f(x, a)$ is asymptotic to $y = c$ for some $c \geq 0$.

Proof. Suppose that $y = f(x, a)$ is neither of type (a) nor (b). Since $f(x, a) - (\partial f/\partial x)(x, a)x$ is never zero, $f(x, a)$ is defined globally on $[1, \infty)$. It follows that $f(x, a) > 0$, $(\partial f/\partial x)(x, a) < 0$, $(\partial^2 f/\partial x^2)(x, a) > 0$ on $[1, \infty)$. Since $y = f(x, a)$ does not intersect with the $x$-axis, $\lim_{x \to \infty}(\partial f/\partial x)(x, a) = 0$. When $a < 0$, by (2) and $\lim_{x \to \infty}(\partial f/\partial x)(x, a) = 0$, we see that the $x$-axis
is the asymptotic line of $y = f(x, a)$. When $a = 0$, $y = f(x, a)$ is asymptotic to $y = c$ for some $c \geq 0$.

**Proposition 2.** $y = f(x, 0)$ is of type (a).

**Proof.** When $a = 0$, (2) becomes

$$\frac{d^2 y}{dx^2} = \left\{ 1 + \left( \frac{dy}{dx} \right)^2 \right\} \frac{dy}{dx} \left( y' \frac{dy}{dx} - x \right)^{-1}. \tag{3}$$

First we show that $y = f(x, 0)$ is not of type (b). If there is $x_0 > 1$ such that $(\partial f/\partial x)(x_0, 0) = 0$, then $y = f(x_0, 0)$ and $y = f(x, 0)$ are two distinct solutions of (3) with the same initial condition $y(x_0) = f(x_0, 0)$ and $(dy/dx)(x_0) = 0$. This is a contradiction. Next suppose that $y = f(x, 0)$ is of type (c). Then $f(x, 0) > 0$, $-1 < (\partial^2 f/\partial x^2)(x, 0) < 0$ and $(\partial^2 f/\partial x^2)(x, 0) > 0$ on $[1, \infty)$. By (3)

$$\frac{d^2 f}{dx^2} \leq -\frac{1}{x} \left\{ 1 + \left( \frac{df}{dx} \right)^2 \right\} \frac{df}{dx}, \tag{4}$$

where we abbreviate $(\partial f/\partial x)(x, 0) = (df/dx)$, $(\partial^2 f/\partial x^2)(x, 0) = (d^2 f/dx^2)$. We denote the solution of

$$\frac{d^2 y}{dx^2} = -\frac{1}{x} \left\{ 1 + \left( \frac{dy}{dx} \right)^2 \right\} \frac{dy}{dx}, \tag{5}$$

with the initial condition $y(1) = 1$, $(dy/dx)(1) = -1$ as $y = g(x)$. Then we obtain

$$g(x) = -\frac{1}{\sqrt{2}} \log \left( x + \sqrt{x^2 - 1/2} \right) + \frac{1}{\sqrt{2}} \log \left( 1 + \frac{1}{\sqrt{2}} \right) + 1. \tag{6}$$

By the comparison theorem of a first-order ordinary differential equation, we get $df/dx \leq dg/dx$ for all $x \in [1, \infty)$. Integrating this inequality, we get $f(x, 0) \leq g(x)$ for all $x \in [1, \infty)$. Since $\lim_{x \to -\infty} g(x) = -\infty$, $\lim_{x \to -\infty} f(x, 0) = -\infty$. This is a contradiction. The conclusion follows from Lemma 3.

**Proposition 3.** There is $a_0 < 0$ satisfying the following property:

(i) $y = f(x, a)$ is of type (a) for all $a \in (a_0, 0]$.

(ii) $\lim_{a \to a_0^+} x_a = \infty$, where $x_a$ is the $x$-coordinate of the intersection of $y = f(x, a)$ with the $x$-axis.

(iii) $y = f(x, a_0)$ is of type (c).

**Proof.** We set

$$A = \{ a \in (-\infty, 0] : y = f(x, a) \text{ is of type (a) for all } a \in [a, 0] \}. \tag{7}$$

By Proposition 2, Lemma 3 and the continuity on the parameter of the solution of an ordinary differential equation, we see that $A$ is a nonempty connected
open subset of \((-\infty, 0]\). Set \(A = (a_0, 0]\), where \(-\infty \leq a_0 < 0\). For every \(a \in (a_0, 0]\), \(0 \leq f(x, a) \leq 1\), \(-1 \leq (\partial f/\partial x)(x, a) < 0\) and \((\partial^2 f/\partial x^2)(x, a) > 0\) on \([1, x_a]\). So we get from (2)

\[
\frac{\partial^2 f}{\partial x^2}(x, a) \geq -\frac{a}{4} f(x, a) \quad \text{on} \quad [1, x_a].
\]

We denote by \(h(x, a)\) the solution of \(dy^2/dx^2 = -ay/4\) satisfying the initial condition \(y(1) = 1\), \((dy/dx)(1) = -1\). We have

\[
h(x, 0) = 2 - x,
\]

\[
h(x, a) = \frac{1}{2} \left( 1 - \frac{2}{\sqrt{-a}} \right) \exp \left( \frac{\sqrt{-a}(x - 1)}{2} \right) + \frac{1}{2} \left( 1 + \frac{2}{\sqrt{-a}} \right) \exp \left( -\frac{\sqrt{-a}(x - 1)}{2} \right), \quad (a < 0).
\]

When \(-4 < a < 0\), \(y = h(x, a)\) intersects with the \(x\)-axis at \(x = (-a)^{-1/2}\).

\[
\log((2 + \sqrt{-a}/(2 - \sqrt{-a}))) + 1.
\]

For \(a \in (a_0, 0]\), we set

\[
B_a = \{x \in (1, x_a]; f(t, a) > h(t, a), \frac{\partial f}{\partial x}(t, a) > \frac{\partial h}{\partial x}(t, a) \quad \text{for all} \quad t \in (1, x_a)\}.
\]

We can easily that \(\sup B_a = x_a\). Thus for \(a \in (a_0, 0]\), \(f(x, a) \geq h(x, a)\) on \([1, x_a]\). Therefore \(x_a \geq (-a)^{-1/2}(2 + \sqrt{-a}/(2 - \sqrt{-a}))) + 1\) for all \(a \in (a_0, 0]\). Since \(\lim_{a \to -4+0}(-a)^{-1/2}(2 + \sqrt{-a}/(2 - \sqrt{-a}))) + 1 = \infty\), we get \(a_0 > -4\) and \(\lim_{a \to -4+0} x_a = \infty\). From (i) and (ii) we get (iii).

**Theorem.** There is an \(O(2) \times O(2)\)-invariant complete hypersurface with constant negative scalar curvature in \(E^4\).

**Proof.** By Proposition 3, there is \(a_0 < 0\) such that \(y = f(x, a_0)\) \((x \geq 1)\) is of type (c). Since the equations (1), (2) are symmetric with respect to \(x\) and \(y\), we can extend \(y = f(x, a_0)\) naturally on \((0, \infty)\) to get a global solution \(y\) of (1). Since \(y\) has an infinite length, \(M_y\) is complete.

**Corollary.** There is a complete hypersurface of constant negative scalar curvature in \(E^n\) for \(n \geq 4\).

**Remarks.** (1) By numerical analysis, we see that the scalar curvature of \(M_y\) constructed in the proof of the theorem is about -0.52.

(2) It seems that except \(S^3(\mathbb{R})\) and \(S^1(\mathbb{R}) \times E^2\), \(M_y\) in the theorem is the only \(O(2) \times O(2)\)-invariant complete hypersurface with constant scalar curvature in \(E^4\) (up to homothety). However, it seems that there are many \(O(p) \times O(q)\)-invariant complete hypersurfaces with constant scalar curvature in \(E^{p+q}\) for \(p + q > 4\). These problems will be studied in the future.
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REFERENCES


Department of Mathematics, Faculty of Science, Hirosaki University, Hirosaki, 036, Japan