MINIMAL HARMONIC FUNCTIONS ON DENJOY DOMAINS

STEPHEN J. GARDINER

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Abstract. Let $\Omega = \mathbb{R}^n \backslash E$, where $E$ is a closed subset of the hyperplane \{x_n = 0\} and every point of $E$ is regular for the Dirichlet problem on $\Omega$. Further, let $\alpha_k$ denote the $(n-1)$-dimensional measure of the set $\{X \in \Omega: x_n = 0, \, e^k < |X| < e^{k+1}\}$. It is known that the cone, $\mathcal{P}_E$, of positive harmonic functions on $\Omega$ which vanish on $E$ has dimension 1 or 2. In this paper it is shown that if $\sum e^{-\alpha_k} e^{n/(n-1)} < +\infty$, then $\dim \mathcal{P}_E = 2$. This result, which in the case $n = 2$ implies a recent theorem of Segawa, is also shown to be sharp.

1. Introduction and results

Points of $\mathbb{R}^n$ $(n \geq 2)$ are denoted by $X = (X', x_n)$, where $X' \in \mathbb{R}^{n-1}$. We call $\Omega$ a Denjoy domain if $\Omega = \mathbb{R}^n \backslash E$, where $E$ is a nonempty closed proper subset of the hyperplane \{x_n = 0\} such that each point of $E$ is regular for the Dirichlet problem in $\Omega$.

Let $\mathcal{P}_E$ denote the cone of positive harmonic functions on $\Omega$ which vanish on $E$. It is known (see [1] or [2]) that either all functions in $\mathcal{P}_E$ are proportional or $\mathcal{P}_E$ is generated by two linearly independent minimal harmonic functions. (A positive harmonic function $u$ on $\Omega$ is called minimal if any other positive harmonic function $v$ on $\Omega$ satisfying $v \leq u$ is proportional to $u$.) We describe these cases by writing $\dim \mathcal{P}_E = 1$ and $\dim \mathcal{P}_E = 2$ respectively.

Roughly speaking, $\dim \mathcal{P}_E = 2$ if the set \{x_n = 0\} \backslash E is "sufficiently sparse near infinity". Several results in this direction can be found in Benedicks [2]. Recently Segawa [4], working in the complex plane, added the following.

**Theorem A.** Let $n = 2$. If there exists $\lambda > \frac{1}{2}$ such that

$$
\int_{\{x: (x,0) \in \Omega, |x| \geq t\} \times dx = O(t^{-1} [\log t]^{-\lambda}) \quad (t \to +\infty),
$$

then $\dim \mathcal{P}_E = 2$.

(In fact, Segawa stated his result for harmonic functions with pole at the origin rather than at infinity. The above is an equivalent formulation based on inversion in the unit circle.)

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The purpose of this paper is to provide a result of this type for \( \mathbb{R}^n \), which in the case \( n = 2 \) improves Theorem A. We will also show that our condition is sharp.

For each \( k \in \mathbb{N} \) let

\[
F_k = \{ X' \in \mathbb{R}^{n-1} : (X',0) \in \Omega \text{ and } e^k < |X'| < e^{k+1} \},
\]

and put \( \alpha_k = |F_k| \), the \((n-1)\)-dimensional Lebesgue measure of \( F_k \). Our main result is as follows.

**Theorem 1.** If \( \sum e^{-nk} \alpha_k^{n/(n-1)} < +\infty \), then \( \dim \mathcal{P}_E = 2 \).

The proof of Theorem 1 relies on the work of Benedicks mentioned above and a balayage-type argument. Details can be found in §2. To see that Theorem 1 includes Theorem A, note that if \( n = 2 \) and (1) hold, then

\[
e^{-k} \alpha_k \leq e^{2+k} \int_{F_k} x^{-2} \, dx = O(k^{-k}) \quad (k \to \infty),
\]

so \( \sum e^{-2k} \alpha_k^{-2} < +\infty \).

In the case \( n \geq 3 \) it is known [2, Corollary 1] that if \( E \) omits an infinite \((n-1)\)-dimensional circular cone, then \( \dim \mathcal{P}_E = 1 \). The following result shows that it is sufficient for \( E \) to omit a certain sequence of \((n-1)\)-dimensional balls. It also establishes the sharpness of Theorem 1.

**Theorem 2.** Let \( (r_k) \) be a sequence of nonnegative real numbers such that \( \sum e^{-nk} r_k^n = +\infty \). If

\[
E = \{ x_n = 0 \} \setminus \bigcup_{k=1}^{\infty} \{(X',0) : |X' - (2e^k,0,\ldots,0)| < r_k \},
\]

then \( \dim \mathcal{P}_E = 1 \).

The proof of Theorem 2 is given in §3.

2. **Proof of Theorem 1**

2.1. We begin by stating a result of Benedicks on which our proofs rely. For each \( X' \in \mathbb{R}^{n-1} \) let \( K(X') \) be the open cube in \( \mathbb{R}^n \) with center \((X',0)\) and side \( e^{-1}|X'| \), all sides being parallel to the coordinate hyperplanes. Further, let \( \Omega(X') = K(X') \setminus E \) and \( \beta_E(X') \) denote the harmonic measure of \( \partial K(X') \) on \( \Omega(X') \) evaluated at the point \((X',0)\). The result below is drawn from [2, Theorems 3 and 4].

**Theorem B.** The following are equivalent:

(i) \( \dim \mathcal{P}_E = 2 \);

(ii) there exists a function \( u \in \mathcal{P}_E \) such that \( u(X) \geq |x_n| \) on \( \mathbb{R}^n \);

(iii) \( \int_{\{ |X'| \geq 1 \}} |X'|^{1-n} \beta_E(X') \, dX' < +\infty \).
2.2. We now collect together some definitions required in the proof of Theorem 1.

Let $h$ denote the function given by $h(X) = |x_n|$, and let $\sigma_n$ be the surface area of the unit sphere in $\mathbb{R}^n$. Also, let

$$F_0 = \{X' \in \mathbb{R}^{n-1}: (X',0) \in \Omega \text{ and } |X'| < e\}$$

and $\alpha_0 = |F_0|$. For each $k \in \mathbb{N}$, we let

$$\gamma_k = \sum_{j=1}^{4} \alpha_{4k-j}$$

and note that the hypothesis of Theorem 1 implies that the series $\sum e^{-4nk} \gamma_{k/n(n-1)}$ converges. We define the open sets

$$G_k = \left(\bigcup_{j=1}^{4} F_{4k-j}\right) \times (-\gamma_{k/n(n-1)}, \gamma_{k/n(n-1)}) \text{ and } U_k = \bigcup_{j=1}^{k} G_j.$$

(If $\gamma_k = 0$, then $G_k$ is empty.)

If $f$ is a function defined on the boundary $\partial W$ of a bounded open set $W$, we use $H[\partial W, f]$ to denote the Perron-Wiener-Brelot solution (if it exists) to the corresponding Dirichlet problem on $W$. (An account of the properties of $H[\partial W, f]$ can be found in [3, Chapter 8].) If $f$ is defined on the hyperplane $\{x_n = 0\}$, we similarly write $I[f]$ for the corresponding half-space Poisson integral; that is,

$$I[f](X) = 2\sigma_n^{-1} \int_{\mathbb{R}^{n-1}} \frac{f(Y',0) \, dY'}{|X' - Y'|^2 + x_n^2)^{n/2}} \quad (x_n > 0).$$

Now suppose that $s$ is a nonnegative subharmonic function on $\mathbb{R}^n$. We define

$$(J_0 s)(X) = \begin{cases} h(X) + I[s] (X', |x_n|) & (x_n \neq 0) \\ s(X) & (x_n = 0) \end{cases}$$

and

$$(J_k s)(X) = \begin{cases} H[U_k, s](X) & (X \in U_k) \\ s(X) & (X \in \mathbb{R}^n \setminus U_k) \end{cases}$$

for any $k \in \mathbb{N}$.

2.3. For any set $A \subseteq \mathbb{R}^{n-1}$, let $\chi_A$ denote the function valued 1 on $A \times \{0\}$ and 0 elsewhere on $\{x_n = 0\}$. Also, let $r_{n-1}$ be the radius of the ball $B$ centered at the origin $O'$ of $\mathbb{R}^{n-1}$ for which $|B| = 1$. We will need the following simple lemma.

**Lemma 1.** There is a positive constant $c_n < 1$, depending only on $n$, such that if $A$ is a measurable subset of $\mathbb{R}^{n-1}$ satisfying $|A| = 1$, then

$$I[\chi_A](X', 1) \leq c_n \quad (X' \in \mathbb{R}^{n-1}).$$
In fact,
\[
I[\chi_A](x', 1) = \frac{2}{\sigma_n} \int_A \frac{dY'}{\{|x' - Y'|^2 + 1\}^{n/2}}
\]
\[
= \frac{2}{\sigma_n} \int_{\{Y' \in A: |x' - Y'| \leq r_{n-1}\}} \frac{dY'}{\{|x' - Y'|^2 + 1\}^{n/2}}
\]
\[
+ \frac{2}{\sigma_n} \sum_{k=1}^{\infty} \frac{r_{n-1}^2}{\{|x' - Y'| > r_{n-1}\}} \frac{dY'}{\{|x' - Y'|^2 + 1\}^{n/2}}
\]
\[
= I[\chi_B](O', 1).
\]

2.4. We claim that it is enough to prove Theorem 1 in the special case where
\[
\alpha_{4k} = 0 \quad \text{for all } k = 0, 1, 2, \ldots.
\]
To see this, let
\[
D = \bigcup_{k=1}^{\infty} \{(x', 1): e^{4k} \leq |x'| \leq e^{4k+1}\},
\]
\[
E_j = D \cup \{(e^j x', 0): (x', 0) \in E\} \quad (j = 0, 1, 2, 3),
\]
and observe that
\[
(3) \quad \beta_{E_j}(x') = \beta_E(x^{-j} x') \quad (e^{4k+2} < |x'| < e^{4k+3}; k = 0, 1, 2, \ldots).
\]

Now assume that the special case of the theorem has been established. Then
we know that \(\dim \mathcal{P}_{E_j} = 2\) for \(j = 0, 1, 2, 3\); and so by Theorem B, (2) holds,
with \(E\) replaced by any \(E_j\). From (3) it follows that
\[
\int \sum_{k=1}^{\infty} \{(x': e^{4k+j} < |x'| < e^{4k+j+1}\} |x'|^{1-n} \beta_E(x') dX' < +\infty
\]
for each \(j\). Hence (2) holds, and by a further application of Theorem B,
\(\dim \mathcal{P}_E = 2\).

In the proof of Theorem 1 we can also assume that the sum \(T\) of the series
\[
\sum e^{-4nk} y_k^{n/(n-1)}
\]
satisfies
\[
(4) \quad c_n + 2T \sigma_n^{-1} (e^{-3} - e^{-7/2})^{-n} < 1,
\]
for the convergence of (2) is unaffected when we replace \(E\) by \(E \cup \{(x', 0): |x'| \leq a\}\) for any \(a > 0\). The left-hand side of (4) will be denoted by \(d_n\).

2.5. In the light of §2.4 it remains to prove Theorem 1 under the additional assumptions that (4) holds and that \(\alpha_{4k} = 0\) for all \(k = 0, 1, 2, \ldots\).

Let \(k \in \mathbb{N}\) and suppose that \(s\) is a nonnegative subharmonic function on \(\mathbb{R}^n\) which vanishes on \(E \cup \{(x', 0): |x'| \geq e^{4k-4}\}\). Suppose also that the \(\delta\)-subharmonic function \(s - \delta\) is bounded above on \(\mathbb{R}^n\). Then the function \(S = J_k s\) is also nonnegative and subharmonic on \(\mathbb{R}^n\) (the regularity of \(U_k\) for the Dirichlet problem follows from the regularity of \(\Omega\)); we have \(S \geq s\); the
function $S - h$ is bounded above; and $S$ vanishes on $E \cup \{(X', 0) : |X'| \geq e^{4k}\}$. A Phragmén-Lindelöf argument applied to each half-space now shows that $J_0 S \geq S$ on $\{X : x_n \neq 0\}$ and thus that $J_0 S$ is subharmonic on $\mathbb{R}^n$. Again $J_0 S - h$ is bounded above and $J_0 S$ vanishes on $E \cup \{(X', 0) : |X'| \geq e^{4k}\}$.

It follows that the functions $(s_k)$ defined inductively by

$$s_0 = h; \quad s_{2k-1} = J_k s_{2k-2}; \quad s_{2k} = J_0 s_{2k-1}$$

are all nonnegative and subharmonic on $\mathbb{R}^n$, vanish on $E$, and form an increasing sequence.

2.6. To prove that $\lim_{k \to \infty} s_k$ is finite, we establish the following lemma.

**Lemma 2.** Let $k \in \mathbb{N}$. If there is a positive constant $C$ such that

$$(5) \quad s_{2k-1}(X', 0) \leq C \gamma_j^{1/(n-1)} \quad ((X', 0) \in G_j ; j \in \mathbb{N}),$$

then

$$s_{2k+1}(X', 0) \leq (1 + C d_n) \gamma_j^{1/(n-1)} \quad ((X', 0) \in G_j ; j \in \mathbb{N}).$$

To see this, suppose (5) holds, let $\gamma_j \neq 0$, and let

$$K = \{X : e^{(8j-7)/2} < |X'| < e^{(8j+1)/2} \text{ and } |x_n| < \gamma_j^{1/(n-1)}\}.$$ 

Using the fact that $\alpha_{4j-4} = 0 = \alpha_{4j}$, we have, for any $X \in \partial K$,

$$2x_n \sigma_n^{-1} \int_{\{Y' \in \mathbb{R}^{n-1} : (Y', 0) \not\in K\}} \frac{s_{2k-1}(Y', 0) dY'}{\{|X' - Y'|^2 + x_n^2\}^{n/2}}$$

$$\leq 2 \sigma_n^{-1} \gamma_j^{1/(n-1)} C \left\{ \left| e^{(8j-7)/2} - e^{4j-4} \right|^{n-1} \sum_{i=1}^{j-1} \gamma_i^{n/(n-1)} 
\quad + \sum_{i=j+1}^k \left| e^{4i-3} - e^{(8j+1)/2} \right|^{n} \gamma_i^{n/(n-1)} \right\}$$

$$\leq 2 \sigma_n^{-1} \gamma_j^{1/(n-1)} C \left\{ \left| e^{1/2} - 1 \right|^{n} \sum_{i=1}^{j-1} e^{-4ni} \gamma_i^{n/(n-1)} 
\quad + \left| e^{-3} - e^{-7/2} \right|^{n} \sum_{i=j+1}^k e^{-4ni} \gamma_i^{n/(n-1)} \right\}$$

$$\leq 2 \sigma_n^{-1} \gamma_j^{1/(n-1)} C [e^{-3} - e^{-7/2}]^{n} T,$$

where $T$ is as defined in §2.4. Also, using Lemma 1 and the dilation $X \mapsto \gamma_j^{1/(n-1)} X$, we have

$$2x_n \sigma_n^{-1} \int_{\{Y' \in \mathbb{R}^{n-1} : (Y', 0) \not\in K\}} \frac{s_{2k-1}(Y', 0) dY'}{\{|X' - Y'|^2 + x_n^2\}^{n/2}}$$

$$\leq \gamma_j^{1/(n-1)} C \max\{c_n, 2 \sigma_n^{-1} [e^{-3} - e^{-7/2}]^{n} e^{-4nj} \gamma_j^{n/(n-1)} \}$$

$$= \gamma_j^{1/(n-1)} C c_n \quad (X \in \partial K).$$
(The last step is possible since, by the final paragraph of §2.4, we can assume \( j \) to be sufficiently large to achieve this.) Hence, if \( X \in \partial K \),

\[
s_{2k}(X) = (J_0s_{2k-1})(X) \\
\leq \gamma_j^{1/(n-1)}\{1 + C[2T_0^{-1}(e^{-2} - e^{-7/2})^{-n} + c_n]\} \\
= \gamma_j^{1/(n-1)}(1 + C\delta_n),
\]

where \( \delta_n \) is as defined in §2.4. It follows that, if \( (X',0) \in G_j \), then

\[
s_{2k+1}(X', 0) = (J_{k+1}s_{2k})(X', 0) \\
= \begin{cases} 
H[G_j, s_{2k}](X', 0) & (j \leq k + 1) \\
0 & (j > k + 1)
\end{cases}
\]

\[
\leq \max\{s_{2k}(Y) : Y \in \partial K\} \\
\leq \gamma_j^{1/(n-1)}(1 + C\delta_n),
\]

the penultimate step being a consequence of the maximum principle and the fact that \( G_j \subseteq K \). The lemma is now established.

2.7. We now complete the proof of Theorem 1. Since

\[
s_1(X', 0) = (J_1h)(X', 0) \left\{ \begin{array}{ll}
\leq \gamma_1^{1/(n-1)} & ((X', 0) \in G_1) \\
= 0 & ((X', 0) \notin G_1)
\end{array} \right.
\]

it follows from Lemma 2 that

\[
s_{2k+1}(X', 0) \leq (1 + \delta_n + \delta_n^2 + \cdots + \delta_n^k)\gamma_j^{1/(n-1)} \quad ((X', 0) \in G_j; j \in \mathbb{N}).
\]

Since \( \delta_n < 1 \), it follows that the function \( s = \lim_{k \to \infty} s_k \) is locally bounded on \( \{x_n = 0\} \). Further, \( s_{2k-1} \) is harmonic in \( U_k \), so \( s = \lim s_{2k-1} \) is harmonic in \( \bigcup_j G_j \). We can assume this latter set to be nonempty (otherwise the theorem is vacuously true), so there are points of the half-spaces \( \{x_n > 0\} \) and \( \{x_n < 0\} \) where \( s \) is finite. Since \( s_{2k} \) is harmonic in \( \{x_n \neq 0\} \) for each \( k \), so is \( s = \lim s_{2k} \). In fact, applying the monotone convergence theorem to the equation

\[
s_{2k} = J_0s_{2k-1},
\]

we find that

\[
s(X) = h(X) + I[s](X', |x_n|) \quad (x_n \neq 0),
\]

and so \( s \) is locally bounded on \( \mathbb{R}^n \).

We have now established that \( s \) is harmonic on the set \( \mathbb{R}^n \setminus (E \cup L) \), where

\[
L = \{(X', 0) : \log |X'| \in \mathbb{N}\}.
\]

Since \( L \) is polar and \( s \) is locally bounded on \( \mathbb{R}^n \), it follows that \( s \) can be redefined (if necessary) on \( L \cap E \) in such a way that \( s \) is harmonic on \( \mathbb{R}^n \setminus E \).

Now let \( a > 0 \) and \( B = \{X \in \Omega : |X| < a\} \). Since each \( s_k \) is subharmonic on \( \mathbb{R}^n \), we have \( s_k \leq H[B, s_k] \) on \( B \), whence \( s \leq H[B, s] \) on \( B \) by monotone convergence. Since, further, each \( s_k \) vanishes on \( E \), it follows that \( s \) continuously vanishes on the set \( \{X \in E : |X| < a\} \). (Any point of the latter set is
regular for the Dirichlet problem on \( B \). Thus, as \( a \) can be arbitrarily large, \( s \) continuously vanishes on \( E \). Also, \( s \geq s_0 = h \). The equivalence of (i) and (ii) in Theorem B now shows that \( \dim \mathcal{P}_E = 2 \).

3. Proof of Theorem 2

Let \( E' = \{(X', 0): |X'| \geq 1\} \) and let \( u \in \mathcal{P}_{E'} \) be symmetric in \( x_n \). Clearly there is a positive constant \( c \) such that

\[
u(X) = c|x_n| + I[u](X', |x_n|) \quad (x_n \neq 0)
\]

and so, multiplying by a suitable factor, we can assume that \( u(X) \leq 1 + |x_n| \) for all \( X \in \mathbb{R}^n \).

Let \((r_k)\) and \( E \) be as in the statement of Theorem 2 and let \( 2\rho_k = \min\{r_k, e^k\} \). Clearly \( \sum e^{-n_k} \rho_k^n = +\infty \). Let

\[
B_k = \{X' \in \mathbb{R}^{n-1}: |X' - (2e^k, 0, \ldots, 0)| < \rho_k\}.
\]

If \( X' \in B_k \), then \( \beta_E(X') \geq \beta_s(X') \), where \( \beta_s(X') \) is the harmonic measure of \( \partial K(X') \) on the set

\[
W = K(X') \backslash \{(Y', 0): |Y' - X'| \geq \rho_k\}
\]

evaluated at the point \((X', 0)\). Applying the maximum principle to \( W \), it follows that

\[
\beta_s(X') \geq \left(1 + \frac{|X'|}{2e\rho_k}\right)^{-1} u(0),
\]

whenever \( \rho_k \neq 0 \). Hence

\[
\int_{\{X'| \geq 1\}} |X'|^{1-n} \beta_E(X') \, dX' \geq u(0) \sum_{\rho_k \neq 0} \int_{B_k} |X'|^{1-n} \left(1 + \frac{|X'|}{2e\rho_k}\right)^{-1} \, dX'
\]

\[
\geq \frac{c_{n-1}}{(n-1)}u(0) \sum_{\rho_k \neq 0} e^{(k+1)(1-n)} \rho_k^{n-1} \left(1 + \frac{e^k}{2\rho_k}\right)^{-1}
\]

\[
\geq \frac{c_{n-1}}{(n-1)}u(0) e^{1-n} \sum_{k=1}^{\infty} \rho_k e^{-nk} \rho_k^n
\]

\[= +\infty.
\]

It now follows from Theorem B that \( \dim \mathcal{P}_E = 1 \).

References

2. M. Benedicks, Positive harmonic functions vanishing on the boundary of certain domains in \( \mathbb{R}^n \), Ark. Mat. 18 (1980), 53-71.


Department of Mathematics, University College, Dublin 4, Ireland