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# A CLASS OF SIMPLE LIE ALGEBRAS OF CHARACTERISTIC THREE

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ABSTRACT. We show the existence of a certain infinite class of simple Lie algebras of characteristic three. These algebras, although of neither classical nor Cartan type, resemble algebras of Cartan type in their relationship to each other.

## 1. INTRODUCTION

The conjecture that every finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 7$ is either of classical or of Cartan type has recently been verified in the case of restricted algebras [1]. However for some lower characteristics, including $p = 3$, examples are known of simple algebras of neither type (e.g. in [2], [5], [6]) but with many structural properties generally associated with one or the other of these types. In this paper we construct a class of simple Lie algebras of characteristic three whose members' relationship to each other resembles that of members of a class of algebras of Cartan type in the respect that they can be described as being members of a chain of subalgebras of a certain graded infinite-dimensional algebra of derivations of a divided power algebra. With one exception, an algebra discovered by M. Frank [5], these algebras are not restricted.

In the following section we construct an infinite-dimensional Lie algebra $T(3)$ and note some of its properties. Then in the final section we define algebras $T(3{:}\,n)$ as certain subalgebras of $T(3)$ and show their simplicity as well as their nonisomorphism with algebras of type $W$ of the same dimension.

## 2. THE ALGEBRA $T(3)$

Let $F$ be a field of characteristic three. Let $\mathfrak{A}(m)$ denote the infinite-dimensional commutative associative algebra over $F$ consisting of all formal sums $\sum a_\alpha x^\alpha$ with $\alpha$ ranging over all $m$-tuples of nonnegative integers and having multiplication determined by $x^\alpha x^\beta = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta}$, where $\binom{\gamma}{\alpha} = \binom{\gamma(1)}{\alpha(1)} \cdots \binom{\gamma(m)}{\alpha(m)}$. Let $\mathfrak{A}(m)_j (\mathfrak{A}(m)_{[j]})$ denote all formal sums $\sum a_\alpha x^\alpha$ with $\sum_{i=1}^m \alpha(i) \geq j (= j)$. $\mathfrak{A}(m)$ has a topological grading $\sum_{j\geq 0} \mathfrak{A}(m)_{[j]}$. By [7]

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there exists a unique sequence of continuous mappings $y \to y^{(r)}$ from $\mathfrak{A}(m)_1$ into $\mathfrak{A}(m)$ such that for $x, y \in \mathfrak{A}(m)_1$ and $r$ a natural number $x^{(0)} = 1$, $(x^{\alpha})^{(r)} = ((r\alpha)!/(\alpha!)^r r!)x^{r\alpha}$, $(ax)^{(r)} = a^r x^{(r)}$, and $(x+y)^{(r)} = \sum_{i=0}^{r} x^{(i)} y^{(r-i)}$, where $\alpha! = \prod_{i=1}^{m} \alpha(i)!$. For $\mathbf{n} = (n_1, \dots, n_m)$ an $m$-tuple of positive integers, let $\mathfrak{A}(m{:}\,\mathbf{n})$ denote the subalgebra of $\mathfrak{A}(m)$ spanned by those $x^{\alpha}$ with $\alpha(i) < 3^{n_i}$ for all $i$, and let $|n| = n_1 + \cdots + n_m$. Let $D_i$ be the derivation of $\mathfrak{A}(m)$ given by $D_i(x^{\alpha}) = x^{\alpha - \varepsilon_i}$, where $\varepsilon_i(j) = \delta_{ij}$ and $x^{\beta} = 0$ if $\beta(j) < 0$ for some $j$. Then $W(m) = \{\sum_{i=1}^{m} u_i D_i | u_i \in \mathfrak{A}(m)\}$ is the Lie algebra of special derivations of $\mathfrak{A}(m)$, i.e. continuous derivations $\mathscr{D}$ with $\mathscr{D}(u^{(r)}) = u^{(r-1)}\mathscr{D}(u)$ for all natural numbers $r$ and all $u \in \mathfrak{A}(m)_1$. $W(m{:}\,\mathbf{n}) = \{\sum_{i=1}^{m} u_i D_i | u_i \in \mathfrak{A}(m{:}\,\mathbf{n})\} = \{\mathscr{D} \in W(m) | \mathscr{D}\mathfrak{A}(m{:}\,\mathbf{n}) \subseteq \mathfrak{A}(m{:}\,\mathbf{n})\}$ is a subalgebra of $W(m)$.

Set $x_i = x^{\varepsilon_i}$. Thus $x_i^{(k)} = x^{k\varepsilon_i}$ for $k \geq 0$. Also let $x_i^{(k)} = x^{k\varepsilon_i} = 0$ for $k < 0$. We denote by $K(3)$ the subalgebra of $W(3)$ consisting of those elements $\mathscr{D} \in W(3)$ for which $\mathscr{D}\omega = u\omega$, where $u \in \mathfrak{A}(3)$, $\omega = dx_3 + x_1 dx_2 - x_2 dx_1$, and $\mathscr{D}(hdg) = (\mathscr{D}h)dg + hd(\mathscr{D}g)$. The notation $D_K(f)$ for $f \in \mathfrak{A}(3)$ can be used for elements of $K(3)$, where $D_K(f) = (-D_2 f + x_1 D_3 f)D_1 + (D_1 f + x_2 D_3 f)D_2 + (2f - x_1 D_1 f - x_2 D_2 f)D_3$ (cf. (1.3.1) of [1] noting the different notation in the definition of $\omega$). Then $[D_K(f), D_K(g)] = D_K(-(D_3 g)(f + x_1 D_1 f + x_2 D_2 f) + (D_3 f)(g + x_1 D_1 g + x_2 D_2 g) + (D_1 f)(D_2 g) - (D_2 f)(D_1 g))$. For $k \in \mathbf{Z}$ let $a_{2k} = D_K(x_2^{(2)} x_3^{(k)})$, $b_{2k} = D_K(x_1^{(2)} x_3^{(k)})$, $c_{2k} = D_K(x_1 x_2 x_3^{(k)})$, $d_{2k} = D_K(-x_3^{(k+1)} + x_1^{(2)} x_2^{(2)} x_3^{(k-1)})$, $e_{2k+1} = D_K(x_1 x_2^{(2)} x_3^{(k)} + x_2 x_3^{(k+1)})$, $f_{2k+1} = D_K(-x_1^{(2)} x_2 x_3^{(k)} + x_1 x_3^{(k+1)})$.

Recall that $W(3)$ may be graded by setting $\deg x_i = 1 = -\deg D_i$ for $i = 1, 2$ and $\deg x_3 = 2 = -\deg D_3$. Letting $W(3)_{[j]}$ denote the space of elements of degree $j$, we have $W(3) = \sum_{j \geq -2} W(3)_{[j]}$.

Recall also that $K(3) = \sum_{j \geq -2} K(3)_{[j]}$ is a graded subalgebra of $W(3)$ (with the grading given above). Note that $K(3)_{[-2]} = \langle d_{-2} \rangle$, $K(3)_{[-1]} = \langle e_{-1}, f_{-1} \rangle$, $K(3)_{[0]} = \langle a_0, b_0, c_0, d_0 \rangle$, $a_{2k}, b_{2k}, c_{2k}, d_{2k} \in K(3)_{[2k]}$ for $k \geq 1$ and $e_{2k+1}$, $f_{2k+1} \in K(3)_{[2k+1]}$ for $k \geq 0$.

Define $T(3)_{[j]} = K(3)_{[j]}$ for $j = -2, -1, 0$, and $T(3)_{[2k]} = \langle a_{2k}, b_{2k}, c_{2k}, d_{2k} \rangle$ for $k \geq 1$, $T(3)_{[2k+1]} = \langle e_{2k+1}, f_{2k+1} \rangle$ for $k \geq 0$. Define $T(3) = \sum_{j \geq -2} T(3)_{[j]}$. Also, for $j \geq 1$ define

$$L_{[j]} = \{u \in W(3)_{[j]} | (\operatorname{ad} T(3)_{[-1]})^{j-1} u \subseteq T(3)_{[1]}\}.$$

The following lemma lists facts concerning Lie products involving elements of $T(3)$ (including well-known statements about $K(3)$) which follow easily by direct computation.

**Lemma 2.1.** (a) $T(3)_{[-2]} = [T(3)_{[-1]}, T(3)_{[-1]}]$.

(b) $T(3)_{[-1]} = [T(3)_{[0]}, T(3)_{[-1]}]$.

(c) *If* $u \in W(3)_{[j]}$ *where* $j \geq 0$ *and* $[T(3)_{[-1]}, u] = (0)$, *then* $u = 0$.
(d) *If* $u \in K(3)$, *then* $[T(3)_{[-1]}, u] = (0)$ *if and only if* $u \in K(3)_{[-2]}$.
(e) $T(3)_{[1]} = [T(3)_{[0]}, T(3)_{[1]}]$.
(f) $T(3)_{[j]} \supseteq [T(3)_{[-1]}, T(3)_{[j+1]}]$ *for all* $j \geq 1$.
(g) *If* $u \in T(3)_{[1]}$ *and* $[d_{-2}, u] = 0$, *then* $u = 0$.

**Theorem 2.2.** (a) $T(3)_{[j]} = L_{[j]}$ *for all* $j \geq 1$.
(b) $T(3)$ *is a subalgebra of* $W(3)$ (*in fact, of* $K(3)$).

*Proof.* It follows from the Jacobi identity and (a, b, e) of Lemma 2.1 that $\sum_{j=-2}^{0} T(3)_{[j]} + \sum_{j\geq 1} L_{[j]}$ is a subalgebra of $W(3)$. Thus part (b) follows from part (a). By (f) of the lemma we have $T(3)_{[j]} \subseteq L_{[j]}$. Suppose $u \in L_{[j]} \cap \ker(\operatorname{ad} d_{-2})$. Then $(\operatorname{ad} T(3)_{[-1]})^{j-1} u \subseteq T(3)_{[1]} \cap \ker(\operatorname{ad} d_{-2}) = (0)$ (by $(d, g)$ of the lemma), and so $u = 0$ (by (c) of the lemma). Thus ad $d_{-2}$ is injective on $L_{[j]}$ for all $j \geq 1$. Therefore $\dim L_{[2j]} \leq \dim T(3)_{[0]} = 4$ for all $j \geq 1$ and $\dim L_{[2j+1]} \leq \dim T(3)_{[-1]} = 2$ for all $j \geq 0$. In view of the inclusion already noted, this establishes (a).

## 3. A family of simple subalgebras of $T(3)$

For any positive integer $n$ let $T(3{:}\,n) = T(3) \cap W(3{:}\,\mathbf{n})$, where $\mathbf{n} = (1, 1, n)$. Since it is an intersection of subalgebras of $W(3)$, $T(3{:}\,n)$ is a Lie algebra. From $T(3)$ it inherits the structure of a graded algebra $\sum_{j=-2}^{2(3^n-1)} H_{[j]}$, where $H_{[j]} = T(3)_{[j]} \cap T(3{:}\,n)$. Thus $H_{[j]} = T(3)_{[j]}$ for $-2 \leq j < 2(3^n - 1)$, and $H_{[2k]} = \langle a_{2k}, b_{2k}, c_{2k} \rangle$ for $k = 3^n - 1$. Clearly $\dim T(3{:}\,n) = 2 \cdot 3^{n+1}$.

**Theorem 3.1.** $T(3{:}\,n)$ *is simple.*

*Proof.* By (a, c) of Lemma 2.1, any nonzero ideal $I$ of $T(3{:}\,n)$ contains $d_{-2}$, hence also $\sum_{j=-2}^{2(3^n-2)} H_{[j]}$, the image of ad $d_{-2}$. Since $[H_{[0]}, H_{[j]}] = H_{[j]}$ for $j = 2 \cdot 3^n - 3$, $2 \cdot 3^n - 2$, $I = T(3{:}\,n)$, and simplicity is established. □

Following M. Frank [5] we denote by $T$ the subalgebra of $W(3)$ generated (in our notation) by $D_1, D_2, D_3$, and $Q = (-x_1 x_2^{(2)} x_3 + x_1^{(2)} x_3^{(2)}) D_1 - x_1 x_2 x_3^{(2)} D_2 + x_2^{(2)} x_3^{(2)} D_3$, and by $S$ the subalgebra of $T$ generated by $D_2$, $x_1 D_2 - x_2 D_3$, and $x_2^{(2)} D_1 + x_2 x_3 D_2 - x_3^{(2)} D_3$.

**Proposition 3.2.** $T$ *is isomorphic to* $T(3{:}\,1)$.

*Proof.* The isomorphism $\Phi$ from $T(3{:}\,1)$ to $T$ is determined as follows: $\Phi(a_4) = -[D_1[D_1 Q]]$, $\Phi(b_4) = Q$, $\Phi(c_4) = [D_1 Q]$, $\Phi(d_2) = -[D_2[D_2 Q]] - [D_3[D_1 Q]]$, $\Phi(e_3) = -[D_1[D_2 Q]]$, $\Phi(f_3) = -[D_2 Q]$, $\Phi(g_{i-2}) = -[D_3 \Phi(g_i)]$ for $g = a, b, c, d, e, f$. □

For $\Phi$ defined as above $\Phi(H_{[-2]} \oplus H_{[-1]} \oplus H_{[0]} \oplus H_{[1]} \oplus \langle d_2 \rangle) = S$. That $\Phi^{-1}(S)$ (hence also $S$) is isomorphic to the algebra $L(1)$ of Kostrikin [6] follows from comparing elements (after notational adjustment) or, alternatively, by observing that $S$ is simple (by [5]) and the adjoint action of the subalgebra $\langle d_{-2}, d_0, d_2\rangle$ on $\Phi^{-1}(S)$ is that described in Theorem 2 of [4], then using [3] and [4] to identify first the associated Freudenthal triple system and then the algebra. We note that in [6] Kostrikin mentioned that the gradation of $L(1)$ could be prolonged to give other graded algebras.

The dimension of the algebra $W(2\cdot 3^s\colon \mathbf{r})$ is $2\cdot 3^{s+|\mathbf{r}|}$, as is that of $T(3\colon s+|\mathbf{r}|-1)$. Our concluding result shows that these algebras are not isomorphic.

**Theorem 3.3.** *The algebra* $T(3\colon n)$ *is not isomorphic to any algebra* $W(m\colon \mathbf{r})$.

*Proof.* For $n=1$ this was shown in [5]. Therefore we assume $n>1$.

For an algebra $L$, $P(L)=\{l\in L|(\operatorname{ad} l)^3 \text{ is inner}\}$ is a subspace of $L$. We compare $P(T(3\colon n))$ and $P(W(m\colon \mathbf{r}))$.

In $W(m\colon \mathbf{r})$ define $\mathscr{L}_0$ to be the subalgebra spanned by all $uD_j$ with $u\in \mathfrak{A}(m)_1$. For $\mathscr{D}\in\mathscr{L}_0$, since $(\mathscr{D}x)^3=0$ for $x\in\mathfrak{A}(m)_1$, $\mathscr{D}^3$ is a special derivation leaving $\mathfrak{A}(m\colon \mathbf{R})$ invariant. Therefore, since $(\operatorname{ad}\mathscr{D})^3=\operatorname{ad}\mathscr{D}^3$, $\mathscr{L}_0\subset P(W(m\colon \mathbf{r}))$. $(\operatorname{ad} D_i)^3=0$ if $r_i=1$, and $\mathscr{D}\notin P(W(m\colon \mathbf{r}))$ for $\mathscr{D}\in\langle D_j|r_j\neq 1\rangle$. Thus $P(W(m\colon \mathbf{r}))=\mathscr{L}_0\oplus E$, where $E=\langle D_j|r_j=1\rangle$.

By Theorem 2.2 the $p$-mapping associated with the restricted algebra $\sum_{j\geq 0} W(3)_{[j]}$ can be restricted to $\sum_{j\geq 0} T(3)_{[j]}$, thus establishing its restrictedness. Thus $(\sum_{j\geq 0} T(3)_{[j]})\cap\mathscr{L}_0\subseteq P(T(3\colon n))$ for $\mathscr{L}_0$ associated as above with $W(3\colon \mathbf{n})$. Since $(\operatorname{ad} e_{-1})^3=(\operatorname{ad} f_{-1})^3=0$, but clearly $d_{-2}\notin P(T(3\colon n))$, we have $P(T(3\colon n))=\sum_{j=-1}^{2(3^n-1)} H_{[j]}$.

Thus $P(T(3\colon n))$ has codimension 1, but $P(W(m\colon \mathbf{r}))$ has codimension 1 only if $\mathbf{r}$ is a permutation of $\mathbf{r}'=(1,\dots,1,t)$ with $t>1$. For $W(m\colon \mathbf{r}')$, $(\operatorname{ad} D_m)^{3^t}=0$, while for $\mathscr{D}\in T(3\colon n)\backslash P(T(3\colon n))$, $(\operatorname{ad}\mathscr{D})^{3^n-1}\neq 0$. Thus isomorphism of these algebras requires $t\geq n$. However $\dim W(m\colon \mathbf{r}')=m\cdot 3^{t+m-1}>2\cdot 3^{n+1}$ if $t\geq n$ and $m>2$. Thus there is no isomorphism unless $m=2$ and $t=n$.

Suppose that $\Omega\colon T(3\colon n)\to W(2\colon(1,n))=W$ is an isomorphism. Since $[H_{[-1]},H_{[-1]}]=H_{[-2]}\not\subset P(T(3\colon n))$, but $H_{[-1]}\subset P(T(3\colon n))$ and $\dim H_{[-1]}=2$, $\Omega(H_{[-1]})$ has a basis $\{z_1,z_2\}$ with $z_1\in P(W)\backslash\mathscr{L}_0$ and $z_2\in\mathscr{L}_0$ having a nonzero $x_1D_2$-component (with respect to the $\{x_1^{(i_1)}x_2^{(i_2)}D_j\}$-basis). $\Omega(H_{[0]})\subset P(W)$. If $y\in\Omega(H_{[0]})$ but $y\notin\mathscr{L}_0$, then $[z_2 y]\notin P(W)$, a contradiction since $\Omega(H_{[-1]})\subset P(W)$. Therefore $\Omega(H_{[0]})\subset\mathscr{L}_0$. But then $\Omega(H_{[-1]})=[z_2,\Omega(H_{[0]})]\subset\mathscr{L}_0$, a contradiction, and the theorem is established. □

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