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A MEASURING ARGUMENT FOR FINITE GROUPS

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ABSTRACT. We call attention to a simple measuring argument for finite groups. Direct applications of this argument lead to the construction of certain new characteristic subgroups of finite p-groups as well as an easy proof of a generalization, due to Timmesfeld, of the Thompson replacement lemma. Some applications to finite simple groups are also given.

The purpose of this note is to call attention to an elementary observation herein recorded as Lemma 1.1. We consider Lemma 1.1 to represent a "measuring argument", in the sense that Herstein's textbook [2] refers to a "counting argument for finite groups" to describe the well-known result that

$$|AB| = |A|\,|B|/|A \cap B|$$

when A and B are subgroups of a finite group G.

Our measuring argument generates some characteristic subgroups of p-groups, it provides a simple proof of a "replacement theorem" of Timmesfeld, and has proven extremely useful in some problems involving "failure of Thompson factorization" (especially in [1]). Finally, Theorem 2.2 relates the local structure of a simple group to that of its simple subgroups. Although the result is far from best possible, it is the only one of its type that we know, and may be food for further thought.

1. THE BASIC RESULTS

Let G be a finite group acting on a finite group H, and let $\mathscr{S}(G)$ denote the set of all nonidentity subgroups of G. For any positive real number α, put

$$m_\alpha = m_\alpha(G, H) = \operatorname{Sup}\{|A|^\alpha |C_H(A)|\}_{A \in \mathscr{S}(G)}.$$

Further, put

$$\alpha\mathscr{M} = \alpha\mathscr{M}(G, H) = \{A \in \mathscr{S}(G) : |A|^\alpha |C_H(A)| = m_\alpha\}$$

$$\alpha\mathscr{M}^* = \text{the set of maximal members of } \alpha\mathscr{M}, \text{ under inclusion},$$

$$\alpha\mathscr{M}_* = \text{the set of minimal members of } \alpha\mathscr{M}, \text{ under inclusion}.$$

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1.1. **Lemma.** *Let $A, B \in \alpha\mathscr{M}$, and assume that either $A \cap B \neq 1$ or that $m_\alpha \geq |H|$. Then $AB = BA \in \alpha\mathscr{M}$, and $C_H(A \cap B) = C_H(A)C_H(B)$. Further, if $A \cap B \neq 1$ then $A \cap B \in \alpha\mathscr{M}$.*

Proof. Since $A \in \alpha\mathscr{M}$ we have

$$(*) \qquad |A|^\alpha |C_H(A)| \geq |\langle AB\rangle|^\alpha |C_H(AB)| \geq |AB|^\alpha |C_H(AB)|.$$

Then also

$$(**) \qquad \frac{|B|^\alpha}{|A \cap B|^\alpha} = \frac{|AB|^\alpha}{|A|^\alpha} \leq \frac{|C_H(A)|}{|C_H(AB)|} = \frac{|C_H(A)C_H(B)|}{|C_H(B)|} \leq \frac{|C_H(A \cap B)|}{|C_H(B)|},$$

and hence

$$(***) \qquad |B|^\alpha |C_H(B)| \leq |A \cap B|^\alpha |C_H(A \cap B)|.$$

But $B \in \alpha\mathscr{M}$, and the hypothesis that either $A \cap B \neq 1$ or $m_\alpha \geq |H|$ then implies that

$$|B|^\alpha |C_H(B)| \geq |A \cap B|^\alpha |C_H(A \cap B)|.$$

It now follows that all of the inequalities in $(*)$, $(**)$, and $(***)$ are in fact equalities, and this yields the lemma.

The following is an immediate consequence of 1.1.

1.2. **Lemma.** *Each member A of $\alpha\mathscr{M}^* \cup \alpha\mathscr{M}_*$ is a T.I.-subgroup of G; that is, $A \cap A^g = 1$ if $g \in G - N_G(A)$. Moreover, if $m_\alpha \geq |H|$ then $\alpha\mathscr{M}^*$ has a unique member (namely $\langle \alpha\mathscr{M}\rangle$), and if $m_\alpha > |H|$ then $\alpha\mathscr{M}_*$ has a unique member (namely $\cap \alpha\mathscr{M}$).*

1.3. **Lemma.** *Let X be a subgroup of G, and put $\alpha\mathscr{N} = \alpha\mathscr{M}(X, H)$. Let $A \in \alpha\mathscr{M}^*$ and $B \in \alpha\mathscr{N}$. Then either $B \subseteq A$ or $A \cap B = 1$. Moreover, if $m_\alpha(X, H) \geq |H|$ then $B \subseteq A$.*

Proof. If $|AB : A|^\alpha \geq |C_H(A) : C_H(AB)|$ then $|\langle AB\rangle|^\alpha |C_H(AB)| \geq |A|^\alpha |C_H(A)|$, and then $B \subseteq A$ since $A \in \alpha\mathscr{M}^*$. Alternatively we have

$$|B : A \cap B|^\alpha < |C_H(A) : C_H(AB)| \leq |C_H(A \cap B) : C_H(B)|,$$

which yields $|B|^\alpha |C_H(B)| < |A \cap B|^\alpha |C_H(A \cap B)|$. But $B \in \alpha\mathscr{N}$ and $A \cap B \subseteq X$, so we conclude that $A \cap B = 1$ and that $|B|^\alpha |C_H(B)| < |H|$. This proves the lemma.

1.4. **Corollary.** *Let X be a subgroup of G such that $|X|^\alpha |C_H(X)| \geq |H|$, and let $\{A\} = \alpha\mathscr{M}^*$. Then $A \cap X \supseteq \langle \alpha\mathscr{M}(X, H)\rangle$.*

1.5. **Corollary.** *Let p be a prime, and suppose that there exists a nonidentity p-subgroup X of G such that $|X|^\alpha |C_H(X)| \geq |H|$. Let A be the unique member of $\alpha\mathscr{M}^*$. Then p divides $|A|$.*

1.6. **Lemma.** *Assume that $m_\alpha \geq |H|$ and that $|\alpha\mathscr{M}_*| > 1$ (so that in fact $m_\alpha = |H|$, by 1.2). Let A and B be two distinct members of $\alpha\mathscr{M}_*$. Then $[A, B] = 1$.*

Proof. Set $D = AB$. Then $D \in \alpha\mathscr{M}$ by 1.1, and $|D| = |A|\,|B|$ by 1.2. If both A and B are normal in D than $[A, B] \subseteq A \cap B = 1$, so we may assume by way of contradiction that A is not normal in D.

Let p be a prime dividing $|D|$, and let P be a Sylow p-subgroup of D such that $P \cap A$ is a Sylow p-subgroup of A. We may assume that p is chosen so that P does not normalize A. Let $x \in P - N_P(A)$. Then $D \supseteq AA^x$, of order $|A|^2$, and we have $AA^x \in \alpha\mathscr{M}$. Then $AA^x \cap B \neq 1$, so $B \subseteq AA^x$ and $AA^{\hat{x}} = D$. Thus $|A| = |B|$, so $P \cap A \neq 1$. Since A is a T.I.-subgroup of D we have $N_P(P \cap A) = N_P(A)$. Let $y \in N_P(N_P(A)) - N_P(A)$. Then $D = AA^y$. Also $(P \cap A)^y \leq N_P(A)$ and $(P \cap A) \cap (P \cap A)^y = 1$. Thus $|P \cap A|^2 \leq |N_P(A)| < |P|$. But as $|D| = |A|^2$ we have $|P| = |P \cap A|^2$, a contradiction.

1.7. **Corollary.** *Assume that $m_\alpha \geq |H|$ and that there exists a p-group (resp. an Abelian p-group) in $\alpha\mathscr{M}$. Then there exists a normal p-subgroup of G (resp. a normal Abelian p-subgroup of G) in $\alpha\mathscr{M}$.*

1.8. *Remark.* Let G, H, and α be as above, and put

$$n_\alpha = n_\alpha(G, H) = \operatorname{Sup}\{|A|\,|C_H(A)|^\alpha\}_{A \in \mathscr{S}(G)}.$$

Then $n_\alpha = (m_{1/\alpha})^\alpha$. Therefore, results 1.1 through 1.7 remain true even if we redefine $\alpha\mathscr{M}$ by taking $\alpha\mathscr{M} = \{A \in \mathscr{S}(G) : |A|\,|C_H(A)|^\alpha = n_\alpha\}$ (and with corresponding changes in the definitions of $\alpha\mathscr{M}^*$ and $\alpha\mathscr{M}_*$).

1.9. *Remark.* Suppose we were to recast all of our earlier definitions by replacing $\mathscr{S}(G)$ with a proper subset $\mathscr{S}$ of $\mathscr{S}(G)$. In this case, two members A and B of $\mathscr{S}$ will be said to be *rounded over* $\mathscr{S}$ if both $\langle AB \rangle$ and $A \cap B$ are in $\mathscr{S} \cup \{1\}$. One then obtains versions of Lemmas 1.1 and 1.3 by adding the hypothesis that A and B be rounded over $\mathscr{S}$, and one obtains the following version of 1.2. Namely, if $A \in \alpha\mathscr{M}^* \cup \alpha\mathscr{M}_*$ (defined as a subset of $\mathscr{S}$, now), and $g \in G$ such that A and A^g are rounded over $\mathscr{S}$, then either $A = A^g$ or $A \cap A^g = 1$. Further, in this context, if $m_\alpha \geq |H|$ (resp. $m_\alpha > |H|$) and $A \in \alpha\mathscr{M}^*$ (resp. $A \in \alpha\mathscr{M}_*$) with A and A^g rounded over $\mathscr{S}$, then $A = A^g$.

2. Easy applications

2.1. **Theorem.** *Let G be a finite, non-Abelian simple group. Then $|G| > |A|\,|C_G(A)|$ for any proper nonidentity subgroup A of G. In particular, $|G| > |A|^2$ for any Abelian subgroup A of G.*

Proof. Take $H = G$ and $\alpha = 1$ in the set-up of §1, and put $m := m_1$ and $\mathscr{M} = 1\mathscr{M}$. Then $m \geq |G|\,|Z(G)| = |G|$, and so $|\mathscr{M}^*| = 1$ by 1.2. Set $\{A\} = \mathscr{M}^*$. Then $1 \neq A \trianglelefteq G$ and so $A = G$, and $m = |G|$. Now take $B \in \mathscr{M}_*$. By 1.6 B commutes with any G-conjugate of B not equal to B. Since $G = \langle B^G \rangle$ and G is not a direct product, we get $B = G$. Thus $\mathscr{M}_* = \mathscr{M}^* = \{G\}$, which proves the theorem.

A slight variation on the proof of 2.1 yields the following stronger result.

2.2. **Theorem.** *Let G be a finite non-Abelian simple group, contained as a subgroup in a group H in such a way that $C_H(G) = 1$. Set $\alpha = \ln|H|/\ln|G|$. Then $|A|^\alpha|C_H(A)| < |H|$ for every proper, nonidentity subgroup A of G.*

Proof. Since $|G|^\alpha|C_H(G)| = |H|$ we have $m_\alpha \geq |H|$. As in the proof of 2.1 we then find that $\alpha\mathscr{M}^* = \alpha\mathscr{M}_* = \{G\}$, and this yields the result.

2.3. **Theorem.** *Let G be a finite group, p a prime, and V a faithful, irreducible $GF(p)G$-module. Suppose that for some positive real number α and some subgroup A of G we have $|A|^\alpha|C_V(A)| \geq |V|$. Then $|G|^\alpha \geq |V|$.*

Proof. Take $H = V$ and apply 1.2 in order to get $|\alpha\mathscr{M}^*| = 1$. Without loss, $\alpha\mathscr{M}^* = \{A\}$, and then $A \trianglelefteq G$. As V is faithful and irreducible we have $|C_V(A)| = 1$, so

$$|G|^\alpha \geq |A|^\alpha = |A|^\alpha|C_V(A)| \geq |V|,$$

as desired.

2.4. **Theorem.** *Let G be a finite group and H a finite group on which G acts. Set*

$\mathscr{A}$ = *the set of all nonidentity Abelian subgroups of G,*

$\mathscr{A}_p$ = *the set of all nonidentity Abelian p-subgroups of G, p a fixed prime,*

$\mathscr{A}_p^e$ = *the set of all nonidentity elementary Abelian p-subgroups of G.*

Fix $\mathscr{S} \in \{\mathscr{A}, \mathscr{A}_p, \mathscr{A}_p^e\}$, and fix a positive real number α. Choose $A \in \mathscr{S}$ so that $|A|^\alpha|C_H(A)|$ is as large as possible and, subject to this condition, so that A is as large (resp., as small) as possible. Then

(a) *For any $g \in G$ such that $[A, A^g] = 1$, we have either $A = A^g$ or $A \cap A^g = 1$.*

(b) *If $|A|^\alpha|C_H(A)| \geq |H|$ (resp., $|A|^\alpha|C_H(A)| > |H|$) then A is weakly closed in $C_G(A)$ with respect to G.*

Proof. Apply 1.2 and 1.9, observing that A and A^g are rounded over $\mathscr{S}$ if A^g commutes with A.

3. Characteristic subgroups and p-groups

Let S be a nonidentity finite p-group. For any positive real number α put

$$n_\alpha = \operatorname{Sup}\{|X|\,|C_S(X)|^\alpha\}_{X \in \mathscr{P}(S)}.$$

Then $n_\alpha > |S|^\alpha$, as one sees by taking $X = Z(S)$. Further, put

$$\alpha\mathscr{N} = \{X \in \mathscr{S}(S) : |X|\,|C_S(X)|^\alpha = n_\alpha\},$$
$$M^\alpha(S) = \langle \cup \alpha\mathscr{N} \rangle, \text{ and}$$
$$M_\alpha(S) = \cap \alpha\mathscr{N}.$$

By 1.2 and 1.8, $M^\alpha(S)$ and $M_\alpha(S)$ are members of $\alpha\mathscr{N}$, and evidently they are characteristic subgroups of S. The collection $\{M^\alpha(S), M_\alpha(S)\}_{\alpha>0}$ has some amusing properties, which we will develop in this section.

3.1. **Lemma.** *Let* $X \in \alpha\mathcal{N}$.

(a) *For any subgroup* W *of* S, *we have* $|W: C_W(X)|^\alpha \leq |X: C_X(W)|$.
(b) *If* $\alpha > 1$ *then* $X \subseteq Z(J(S))$.
(c) $X = C_S(C_S(X))$.

Proof. If (a) fails then

$$|X: C_X(W)| < |W: C_W(X)|^\alpha = |C_S(X)W: C_S(X)|^\alpha,$$

and this yields

$$|X|\,|C_S(X)|^\alpha < |C_X(W)|\,|C_S(X)W|^\alpha \leq |C_X(W)|\,|C_S(C_X(W))|^\alpha,$$

which is contrary to $X \in \alpha\mathcal{N}$.

For part (b), let A be an Abelian subgroup of S of maximal order, and take $\alpha > 1$. By (a) with $W = X$, X is Abelian. Maximality of A then yields

$$|A| \geq |C_A(X)X| = |X|\,|C_A(X)|/|A \cap X| \geq |X|\,|C_A(X)|/|C_X(A)|,$$

so $|A: C_A(X)_| \geq |X: C_X(A)|$. Applying (a) with $W = A$ we find that A centralizes X, and (b) follows.

For part (c), just note that

$$|X|\,|C_S(X)|^\alpha \leq |C_S(C_S(X))|\,|C_S(C_S(C_S(X)))|^\alpha,$$

and that since $X \in \alpha\mathcal{N}$ we get equality here.

3.2. **Lemma.**

(a) $n_\alpha = (n_{1/\alpha})^\alpha$.
(b) $C_S(M^\alpha(S)) = M_{1/\alpha}(S)$, *and* $C_S(M_{1/\alpha}(S)) = M^\alpha(S)$.

Proof. Choose $X \in \alpha\mathcal{N}$ and $Y \in (1/\alpha)\mathcal{N}$. Then

$$\begin{aligned}
|X|\,|C_S(X)|^\alpha &= (|X|^{1/\alpha}|C_S(X)|)^\alpha \\
&\leq (|C_S(C_S(X))|^{1/\alpha}|C_S(X)|)^\alpha \\
&\leq (|Y|\,|C_S(Y)|^{1/\alpha})^\alpha = (n_{1/\alpha})^\alpha \\
&\leq (|C_S(C_S(Y))|\,|C_S(Y)|^{1/\alpha})^\alpha \\
&= |C_S(Y)|\,|C_S(C_S(Y))|^\alpha \\
&\leq |X|\,|C_S(X)|^\alpha = n_\alpha.
\end{aligned}$$

Thus (a) holds.

Now (a) implies that $C_S(X) \in (1/\alpha)\mathcal{N}$ and $C_S(Y) \in \alpha\mathcal{N}$. Therefore $C_S(X) \supseteq M_{1/\alpha}(S)$ and $C_S(Y) \subseteq M^\alpha(S)$. Taking $X = M^\alpha(S)$ and $Y = M_{1/\alpha}(S)$ we obtain (b).

3.3. **Lemma.** *If* $0 < \alpha < \beta$ *then* $M^\alpha(S) \subseteq M^\beta(S)$ *and* $M_\alpha(S) \supseteq M_\beta(S)$.

Proof. By 3.1(a) we have

$$|M_{1/\alpha}(S): C_{M_{1/\alpha}(S)}(M^\beta(S))|^\beta \leq |M^\beta(S): C_{M^\beta(S)}(M_{1/\alpha}(S))|,$$

and

$$|M^{\beta}(S): C_{M^{\beta}(S)}(M_{1/\alpha}(S))| \leq |M_{1/\alpha}(S): C_{M_{1/\alpha}(S)}(M^{\beta}(S))|^{\alpha},$$

which implies that $M_{1/\alpha}(S)$ centralizes $M^{\beta}(S)$. Employing 3.2(b), we then have $M^{\beta}(S) \subseteq M^{\alpha}(S)$. Since $0 < 1/\beta < 1/\alpha$ we obtain also $M^{1/\beta}(S) \subseteq M^{1/\alpha}(S)$ and, by 3.2(b) again, $M_{\beta}(S) \supseteq M_{\alpha}(S)$.

3.4. **Proposition.** *Let $\mathscr{S}$ denote the collection of all subgroups of S, and endow $\mathscr{S}$ with the discrete topology. Then the functions f, g from $(0,\infty)$ to $\mathscr{S}$, given by*

$$f: \alpha \mapsto M^{\alpha}(S),$$
$$g: \alpha \mapsto M_{\alpha}(S)$$

are upper semi-continuous. Moreover, the function h from $(0,\infty)$ to $(0,\infty)$, given by

$$h: \alpha \mapsto n_{\alpha}$$

is continuous.

Proof. Let (α, β) be an open interval on which f is constant, and fix $\tau \in (\alpha, \beta)$. Then

$$\begin{aligned} |M^{\beta}(S)|\,|C_S(M^{\beta}(S))|^{\beta} &\geq |M^{\tau}(S)|\,|C_S(M^{\tau}(S))|^{\beta} \\ &\geq |M^{\tau}(S)|\,|C_S(M^{\tau}(S))|^{\tau} \\ &\geq |M^{\beta}(S)|\,|C_S(M^{\beta}(S))|^{\tau}. \end{aligned}$$

Dividing through by the left-most term, we get

$$1 \geq \frac{|M^{\tau}(S)|}{|M^{\beta}(S)|} \cdot \left(\frac{|C_S(M^{\tau}(S))|}{|C_S(M^{\beta}(S))|}\right)^{\beta} \geq \frac{n_{\tau}}{n_{\beta}} \geq |C_S(M^{\beta}(S))|^{\tau-\beta}.$$

As τ approaches β, $|C_S(M^{\beta}(S))|^{\tau-\beta}$ approaches 1, so n_{τ} approaches n_{β} and, using 3.3, we conclude that $M^{\tau}(S) = M^{\beta}(S)$. Thus, f, g, and h are upper semi-continuous, by 3.2(b). But also

$$\begin{aligned} |M^{\tau}(S)|\,|C_S(M^{\tau}(S))|^{\tau} &\geq |M^{\alpha}(S)|\,|C_S(M^{\alpha}(S))|^{\tau} \\ &\geq |M^{\alpha}(S)|\,|C_S(M^{\alpha}(S))|^{\alpha} \\ &\geq |M^{\tau}(S)|\,|C_S(M^{\tau}(S))|^{\alpha}, \end{aligned}$$

and so

$$1 \geq n_{\alpha}/n_{\tau} \geq |C_S(M^{\tau}(S))|^{\alpha-\tau}.$$

Letting τ approach α, we obtain the continuity of h.

4. A REPLACEMENT THEOREM

We provide a simplified proof of a theorem of Timmesfeld [3], which is itself a generalization of the well-known "Thompson replacement theorem."

4.1. **Theorem** (Thompson Replacement). *Let G be a finite group, H an Abelian group admitting action by G, and let A be a nonidentity subgroup of G, such that $|A||C_H(A)| \geq |B||C_H(B)|$ for every nonidentity subgroup B of A. Choose $x \in C_H([A,A])$ and put $M = [x,A]$. Then the following hold:*

(a) *Either $|A||C_H(A)| = |C_A(M)||C_H(C_A(M))| = |C_A(M)||C_H(A)M|$ or $C_A(M) = 1$.*

(b) *If $|A||C_H(A)| \geq |H|$ and both A and H are p-groups, then $C_A(M) \neq 1$.*

Proof. As in the standard argument, we consider the map

$$\begin{aligned} \phi\colon A/C_A(M) &\mapsto M/C_M(A) \\ aC_A(M) &\mapsto [x,a]C_M(A). \end{aligned}$$

We first show that ϕ is injective. For suppose that we are given $a,b \in A$ with $[x,b] \in [x,a]C_M(A)$. Then

$$\begin{aligned} [a,x][x,b] &\in C_M(A) \\ a^{-1}x^{-1}ab^{-1}xb &\in C_M(A) \\ x^{-1}ab^{-1}xba^{-1} &\in C_M(A) \quad \text{(conjugating by } a^{-1}) \\ [x,ba^{-1}] &\in C_M(A). \end{aligned}$$

Hence $[x,ba^{-1},u] = 1$ for any $u \in A$. Applying the Hall identity, we have

$$[x,ba^{-1},u]^{ab^{-1}}[ab^{-1},u^{-1},x]^{u}[u,x^{-1},ab^{-1}]^{x} = 1,$$

and since $x \in C_H([A,A])$ we get $[ab^{-1},u^{-1},x] = 1$. Thus ab^{-1} centralizes $[u,x^{-1}]$. But $[u,x^{-1}] = [x,u]$ as H is Abelian, so ab^{-1} centralizes $[x,u]$ for all $u \in A$, and hence $ab^{-1} \in C_A(M)$. Thus, ϕ is injective.

We now have $|M/C_M(A)| \geq |A/C_A(M)|$, and hence

$$|A||C_M(A)| \leq |C_A(M)||M|,$$

and

$$\begin{aligned} |A||C_H(A)| &\leq |C_A(M)||M||C_H(A)|/|C_M(A)| \\ &\leq |C_A(M)||C_H(A)M| \\ (*) \qquad &\leq |C_A(M)||C_H(C_A(M))|, \end{aligned}$$

from which (a) follows.

Now suppose that $|A||C_H(A)| \geq |H|$ and that A and H are p-groups. Suppose also that $C_A(M) = 1$. Then $(*)$ shows that $|H| = |A||C_H(A)| = |C_H(A)M|$, so $C_H(A)M = H$ and $M \subseteq [H,A] \subseteq [M,A]$. then $M = 1$ and $C_A(M) \neq 1$, for a contradiction.

4.2. **Theorem** (Timmesfeld Replacement). *Let G be a finite group, H an Abelian group admitting action by G, and let A be a nonidentity subgroup*

of G such that $|A||C_H(A)| \geq |B||C_H(B)|$ for every nonidentity subgroup B of A. Set $M = [C_H([A,A]),A]$. Then the following hold:

(a) *Either $|A||C_H(A)| = |C_A(M)||C_H(C_A(M))|$ or $C_A(M) = 1$.*

(b) *If $|A||C_H(A)| \geq |H|$ and both A and H are p-groups, then $C_A(M) \neq 1$.*

Proof. Set $W = C_H([A,A])$ and $U = [W,A]C_H(A)$. Employing the notation developed in section 1, put $m = m_1(A,H)$, $n = m_1(A,U)$, $\mathscr{C} = 1\mathscr{M}(A,H)$, and $\mathscr{D} = 1\mathscr{M}(A,U)$. We have $|A||C_H(A)| = |A||C_U(A)|$ and so $m = n$. In particular, since $A \in \mathscr{C}$ it follows that $\mathscr{D} \subseteq \mathscr{C}$.

For each $x \in W$ set $M_x = [x,A]$ and $A_x = C_A(M_x)$. Suppose that $C_A(M) \neq 1$. Then $A_x \neq 1$, so Theorem 4.1 implies that $A_x \in \mathscr{C}$ and that $C_H(A_x) = C_H(A)M_x$. Thus $C_H(A_x) = C_U(A_x)$ and hence $A_x \in \mathscr{D}$. But $C_A(M) = \cap A_x$, and so $C_A(M) \in \mathscr{D}$ by 1.1. Then $C_A(M) \in \mathscr{C}$ and $C_H(C_A(M)) = C_U(C_A(M)) = U$. This proves (a).

Now suppose that $|A||C_H(A)| \geq |H|$ and that A and H are p-groups. We may assume that $[H,A] \neq 1$. Then $U \neq H$, so $|A||C_H(A)| = |A||C_U(A)| > |U|$. Then $\mathscr{D}$ has a unique minimal element B, by 1.2. By 4.1 we get $A_x \in \mathscr{D}$ for each $x \in W$, so $B \subseteq \cap A_x = C_A(M)$. Thus, $C_A(M) \neq 1$, and this contradiction proves (b).

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