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ORDINARY DIFFERENTIAL EQUATIONS ON CLOSED SUBSETS OF FRÉCHET SPACES WITH APPLICATIONS TO FIXED POINT THEOREMS

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ABSTRACT. The construction and convergence of an approximate solution to the initial value problem $x' = f(t,x)$, $x(0) = x_0$, defined on closed subsets of a Fréchet space is given. Sufficient conditions that guarantee the existence of an approximate solution are analyzed in a relation to the Nagumo boundary condition used in the Banach space case. It is indicated that the Nagumo boundary condition does not guarantee the existence of an approximate solution. Applications to fixed points are given.

I. INTRODUCTION

In the paper [1] the initial value problem

$$x' = f(t,x), \qquad x(0) = z_0, \tag{I.1}$$

was considered on a closed subset D of a Hausdorff locally convex topological vector space (l. c. t. v. s.) X, where $f\colon [0,T]\times D \to X$, $T > 0$, is continuous, locally bounded (for each $z \in D$ there is a nbhd. W of zero in X such that $f([0,T]\times D\cap(z+W))$ is a bounded subset of X) and dissipative or α-Lipschitz. It was shown that the following condition is sufficient (and also necessary when f is bounded) for the existence of an approximate solution.

Condition C1. *For each absolutely* (abs.) *convex* nbhd. V *of zero, there exists a bounded and balanced subset* B *of* X *such that for each* $\varepsilon > 0$, $x \in D$, *and* $t \in [0,T]$ *we can find* $0 < h \le \varepsilon$ *and* $x_h \in D$ *satisfying* $x_h - x - hf(t,x) \in h(V \cap B)$.

The Nagumo boundary condition, which we call (C2), is sufficient for the existence of approximate solutions in normed spaces, but fails to imply existence in the l. c. t. v. s. case.

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Condition C2. *For each* abs. *convex* nbhd. V *of zero,* $\varepsilon > 0$, $x \in D$ *and* $t \in [0, T]$, *we can find* $0 < h \leq \varepsilon$ *and* $x_h \in D$ *satisfying* $x_h - x - hf(t, x) \in hV$.

Phillips [2] was the first to study existence theorems under compactness conditions of the nonlinear term. Later, Dubinsky [3] obtained a similar result while working in Montel spaces. Millionscikov [4] obtained also some existence theorems using Lipschitz-type conditions. Yuasa [5] gave existence theorems via the measure of noncompactness. None of them, however, obtained solutions on general closed subsets of a l. c. t. v. s. X. Finally, R. S. Hamilton [6] in his expository paper on the inverse function theorem of Nash and Moser gives a very interesting example of an existence theorem in a Fréchet space. His concept of a "smooth Banach map" results in a reduction of the original problem to the Banach space case.

In this paper we introduce a condition (CM) which guarantees the existence of an approximate solution on a metrizable l. c. t. v. s. X, at the same time is expressed only in terms of a metric on X. An l. c. t. v. s. (always assumed Hausdorff) X is metrizable if and only if X has a countable base of absolutely convex nbhds. of zero or, equivalently, X has a countable family of seminorms $\{p_n\}$ that generates the locally convex topology of X. We can always assume that $p_n \leq p_{n+1}$, $n \geq 1$. A function $\|\cdot\|: X \to R^+ \cup \{0\}$ given by

$$\|x\| = \sum_{n=1}^{\infty} c_n \frac{p_n(x)}{1 + p_n(x)} \tag{I.2}$$

for $x \in X$, where $c_n > 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, defines a metric on X, called an F-norm of X. Note that $\|\cdot\|$ is not homogenous.

Proposition I.1. *For* $D \subset X$ *and* $f: [0, T] \times D \to X$, *the condition* (C2) *is satisfied if and only if*

$$\liminf_{h \to 0^+} \left[\inf_{z \in D} \left\| \frac{1}{h} \{x + hf(t, x) - z\} \right\| \right] = 0. \tag{I.3}$$

Proof. (C2) implies (I.3) since $\{x: \|x\| \leq r\}$, $r > 0$ is a base of nbhds. of zero in X and $p_n \leq p_{n+1}$. The other implication is obvious. ■

Let us note that since $\|\cdot\|$ is not homogenous, h in (I.3) can not be pulled out in front of the F-norm $\|\cdot\|$.

Suppose X is metrizable l. c. t. v. s., $D \subset X$, and $\|\cdot\|$ is an F-norm on X. Then for $f: [0, T] \times D \to X$, the following condition is closely related to the Nagumo boundary condition.

Condition CM. *For each* $(t, x) \in [0, T] \times D$

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}(x + hf(t, x), D) = 0,$$

where $\operatorname{dist}(z, D) = \inf_{u \in D} \|z - u\|$.

It is clear that (CM) implies conditions (I.3) and (C2). Indeed, $\|\lambda x\| \le |\lambda|\|x\|$ for $|\lambda| \ge 1$ and $x \in X$. It will be shown in §2 that (CM) guarantees the existence of a solution to (I.1) for f with appropriate compactness or dissipativity properties. First, however, we study situations when (CM) is necessary.

Proposition I.2. *Let X be a metrizable* l.c.t.v.s., *$D \subset X$ and $f\colon [0,T]\times D \to X$ with $T > 0$. Then* (C1) *implies* (CM) *for some F-norm on X.*

Proof. Let $\varepsilon > 0$ and $V_n = \{x\colon p_n(x) \le \varepsilon/n\}$. For each $n \ge 1$ we can find a bounded abs. convex subset $B_n \subset B_{n+1}$ of X such that for each $x \in D$ and $t \in [0,T]$ there are sequences $\{h_n\} \subset \mathbf{R}$ and $\{x_n\} \subset D$ with $0 < h_n \to 0$, $z_n = 1/h_n\{x_n - x - h_n f(t,x)\} \in V_n \cap B_n$, and $z_n \to 0$. If $K_n = \sup p_n(B_n) < \infty$, $c_n > 0$ with $\sum_{n=1}^{\infty} c_n < \infty$, and $M_n = \max\{K_n, c_n\}$, then

$$\frac{1}{h_n}\sum_{k=1}^{\infty}\frac{c_k}{1+M_k}\frac{p_k(h_n z_n)}{1+p_k(h_n z_n)}$$
$$\le \frac{1}{h_n}\sum_{k=1}^{n}\frac{c_k}{1+M_k}p_k(h_n z_n) + \sum_{k=n+1}^{\infty} c_k\frac{p_k(z_n)}{1+p_k(z_n)} = I_1 + I_2.$$

Now, since $p_k \le p_{k+1}$ and $\sum_{k=1}^{\infty} c_k < \infty$, we can find n_0 such that for $n \ge n_0$ we have $I_1 \le \varepsilon/2$ and $I_2 \le \varepsilon/2$. Hence (CM) is satisfied with an F-norm defined by $\sum_{n=1}^{\infty} \frac{c_n}{1+M_n}\frac{p_n(x)}{1+p_n(x)}$. ■

Condition (CM) is necessarily satisfied when (I.1) has solutions in D. Indeed, we have

Proposition I.3. *Let X be a metrizable* l.c.t.v.s., *$D \subset X$, and $f\colon [0,T] \times D \to X$ a bounded and continuous function. If* (I.1) *has a local solution for each $z \in D$, then there exists an equivalent F-norm on X such that* (CM) *is satisfied.*

Proof. Since X is metrizable, we can find (See [7], Prop. III.1.8.) a bounded and abs. convex set B such that $f(D) \subset B$ and on $f(D)$ the topologies induced by X and X_B are the same (X_B denotes the vector space spanned by B with the topology induced by the gauge $\|\cdot\|_B$ of B.) Moreover, $B = \bigcap_{n=1}^{\infty} M_n V_n$, where $V_n = \{x\colon p_n(x) \le 1\}$ and $1 \le M_n \to \infty$. For each $0 \le t < T$ and $x \in D$ let $z\colon [t,T) \to D$ with $z(t+h) = x + \int_t^{t+h} f(s,x(s))\,ds$, where $h > 0$ and $t + h < T$. Then

$$x + hf(t,x) - z(t+h) = \int_t^{t+h} [f(t,x) - f(s,x(s))]\,ds \in X_B$$

and converges to zero in the topology of X_B when $h \to 0$. Therefore $\liminf_{h\to 0^+}(1/h)\operatorname{dist}_B(x + hf(t,x), D) = 0$ for all $(t,x) \in [0,T] \times D$, where $\operatorname{dist}_B(z, D)$ denotes the distance from z to D in the normed space $(X_B, \|\cdot\|_B)$. From the definition of B, $p_n(x) \le M_n\|x\|_B$ for all $x \in X_B$. Hence,

$$\liminf_{h\to 0}\frac{1}{h}\operatorname{dist}'(x + hf(t,x), D) \le \left(\sum_{n=1}^{\infty} c_n\right)\liminf_{h\to 0^+}\frac{1}{h}\operatorname{dist}_B(x + hf(t,x), D)$$

for all $(t,x) \in [0,T] \times D$, where $\operatorname{dist}'(z,D) = \inf_{u\in D} \|z-u\|'$ and $\|x\|' = \sum_{n=1}^{\infty} c_n' \frac{p_n(x)}{1+p_n(x)}$ with $c_n' = \frac{c_n}{M_n}$, $M_n \geq 1$. ■

One may observe that if the assumptions of Propositions I.2 are satisfied and, in addition, $f([0,T]\times D) \subset B$ where $\sup_{n\geq 1} \sup_{z\in B} p_n(z) < \infty$, then (CM) holds for any F-norm defined by (I.2).

II. Existence theorems and applications

We have the following result for the existence of an approximate solution (see [8], Theorem 2.2, for the Banach space case):

Proposition II.1. *Let X be a Fréchet space and D a closed subset of X. Assume that*

(1) *$f\colon [0,T]\times D \to X$ is continuous with $M_f \equiv$ Range f bounded.*
(2) (CM) *is satisfied*

Then for each $z \in D$ and $0 < \varepsilon \leq 1$ there exists and ε-approximate solution $x_\varepsilon(t)$ to (I.,1); *i.e., for some $\{t_i\}$ in $[0,T]$ with $t_0 = 0$, $t_i - t_{i-1} \leq \varepsilon$, $i = 1,2,\dots,p$, and $t_p = T$, the following properties are satisfied*:

(i) *$x_\varepsilon(0) = z_0$, $x_\varepsilon(t_{i-1}) \in D$, and $x_\varepsilon(t)$ is linear on $[t_{i-1},t_i]$ for each $i = 1,2,\dots,p$.*
(ii) *if $t \in (t_{i-1},t_i)$ then $x_\varepsilon'(t) - f(t_{i-1},x_\varepsilon(t_{i-1})) \in B(\varepsilon)$, where $B(\varepsilon) = \{x \in X\colon \|x\| \leq \varepsilon\}$.*
(iii) *if $0 < s \leq t \leq T$, $s \in [t_{k-1},t_k]$ and $t \in [t_{m-1},t_m]$, then $x_\varepsilon(t) - x_\varepsilon(s) \in (t-t_{m-1})B(\varepsilon) + B((t_{m-1}-t_k)\varepsilon) + (t_k - s)B(\varepsilon) + (t-s)M_f$.*
(iv) *if t, $s \in [t_{i-1},t_i]$ and $y \in D$ with $y - x_\varepsilon(t_{i-1}) \in B(\varepsilon(t_i - t_{i-1})) + (t_i - t_{i-1})M_f$, then $f(t,y) - f(s,x_\varepsilon(t_{i-1})) \in B(\varepsilon)$.*

Proof. If $x(t)$ is defined on $[0,t_{i-1}]$, $t_{i-1} < T$, and (i)–(iv) are satisfied with $[0,T]$ replaced by $[0,t_{i-1}]$, let δ_i be the supremum of δ with the following properties (we omit ε as a subscript):

(a) $t_{i-1} + \delta \leq T$,
(b) if t, $s \in [t_{i-1}, t_{i-1}+\delta]$ and $y \in D$ with $y - x(t_{i-1}) \in B(\varepsilon\delta) + \delta M_f$, then $f(t,y) - f(s,x(t_{i-1})) \in B(\varepsilon)$.
(c) $\operatorname{dist}(x(t_{i-1}) + \delta f(t_{i-1},x(t_{i-1}))) \leq \frac{1}{2}\delta\varepsilon$.

Since (CM) is satisfied and f is continuous, we have $\delta_i > 0$. Choose $0 < \frac{1}{2}\delta_i \leq h_i < \delta_i \leq \varepsilon$ such that (a)–(b)–(c) are satisfied with δ replaced by h_i.

Now we define $t_i = t_{i-1} + h_i$ and choose $x(t_i) \in D$ such that

$$\|x(t_{i-1}) + (t_i - t_{i-1})f(t_{i-1},x(t_{i-1})) - x(t_i)\| \leq (t_i - t_{i-1})\varepsilon,$$

which is possible by (c). For $t \in [t_{i-1},t_i]$ we define $x(t) = \{x(t_i) - x(t_{i-1})\} \cdot (t-t_{i-1})/(t_i - t_{i-1}) + x(t_{i-1})$. By these definitions and the fact that $\|\lambda x\| \leq |\lambda|\|x\|$ for $|\lambda| \geq 1$ and $x \in X$, it is clear that (i)–(iv) hold in $[0,t_i]$.

To complete the proof we need to show that $t_p = T$. Suppose on the contrary that $t_i < T$ for every $i \geq 1$. Let $r = \lim t_i$. For $j > i$, using (iii) with

$t = t_{m-1} = t_j$ and $s = t_k = t_i$, we obtain $x(t_j) - x(t_i) \in B((t_j - t_i)\varepsilon) + (t_j - t_i)M_f$. This shows that $\{x(t_i)\}$ is Cauchy, hence convergent to $y \in D$. Continuity of f guarantees that there exists $0 < \delta_0 \leq \varepsilon$ and $i_0 \geq 1$ such that (b) is satisfied for $i \geq i_0$. Furthermore, there exists $i_1 \geq i_0$ and $0 < \delta \leq \delta_0$ such that for $i \geq i_1$ we have

$$\begin{aligned}
&\frac{1}{\delta}\operatorname{dist}(x(t_{i-1}) + \delta f(t_{i-1}, x(t_{i-1})), D) \\
&\quad \leq \frac{1}{\delta}\operatorname{dist}(y + \delta f(r, y), D) + \frac{1}{\delta}\|x(t_i) - y\| + \frac{1}{\delta}\|f(t_{i-1}, x(t_{i-1})) - f(r, y)\| \\
&\quad \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.
\end{aligned}$$

Then $h_i \to 0$ implies $\delta_i \to 0$. But $\delta_i > \delta$, which is a contradiction. ■

If $f([0, T] \times D \cap (z + B(r))) \subset M_f$ for some bounded $M_f \subset X$, and (CM) holds only for all $(t, x) \in [0, T] \times D \cap (z + B(r))$, we can obtain an approximate solution $x_\varepsilon(t) \in z + B(r)$ for $0 \leq t \leq T$, where T and $0 < \varepsilon \leq 1$ are such that $B(\varepsilon) + B(\varepsilon) + B(T\varepsilon) + TM_f \subset B(r)$. For the existence of such $T > 0$ the boundedness of M_f is essential.

Corollary II.2. *Let the hypothesis of Prop.* II.1 *be satisfied. Consider a sequence* $\varepsilon_n \to 0^+$ *and corresponding* ε_n*-approximate solutions* $x_n(t)$. *If* $\lim x_n(t) = x(t)$ *for* $t \in [0, T]$, *then* $x(t)$ *is a solution of* (I.1) *on* $[0, T]$.

Let $\{p_n\}$ be a family of seminorms that generate the locally convex topology of X. For $f: D \to X$, where $D \subset X$, we say that f is ω-dissipative ($\omega = \{w_n\}$) if for each $n \geq 1$ there exists $\omega_n \in \mathbf{R}$ such that $p_n(x - y - h(f(x) - f(y))) \geq (1 - h\omega_n)p_n(x - y)$ for x, $y \in D$ and $h > 0$. If $\omega_n = 0$ for all n, then f is called dissipative. This definition agrees with the concept of dissipativeness in normed spaces.

Proposition II.3. *Let* X *be a Fréchet space and* D *a closed convex subset of* X. *Assume that* $f: D \to X$ *is continuous, locally bounded and* ω*-dissipative. Suppose that* (CM) *is satisfied. Then* (I.1) *has a unique solution* $x_z(t)$ *on* $[0, \infty)$ *for each* $z \in D$. *Moreover, for all* $n \geq 1$,

(1) $p_n(x_z'(t)) \leq p_n(f(z))\exp(\omega_n t), \qquad z \in D, \quad t \geq 0.$
(2) $p_n(x_z(t) - x_y(t)) \leq p_n(z - y)\exp(\omega_n t), \qquad z, y \in D, \quad t \geq 0.$

Proof. By Proposition II.1 we obtain an approximate solution in (I.1). Then, standard arguments (see [8] theorem 6.1 for the Banach space case) show the convergence of the approximate solution to the unique solution to (I.1) on $[0, T]$. Next, using Gronwall's inequality for $p_n(x_z(t) - x_z(s))$ we obtain (1) by letting $s \to t^-$. It follows from (1) that $\lim_{t \to T^-} x_z(t)$ exists and is in D, thus x_z can be continued past T. To get (2) it is enough to apply Gronwall's lemma to $m(t) = p_n(x_z(t) - x_y(t))$ for $t \geq 0$, $x, y \in D$. ■

We can now state our main result.

Theorem II.4. *Let X be a Fréchet space, D a closed convex and locally bounded subset of X, and $f: D \to X$ a continuous and locally bounded map which is ω-dissipative with $\omega_n < 1$ for all n. Suppose that* (CM) *is satisfied with f replaced by $f - I$. Then f has a unique fixed point in D.*

Proof. $f - I$ is ω-dissipative with $\omega = \{\omega_n - 1\}$ and $\omega_n - 1 < 0$ for all n. Define $S(t)z = x_z(t)$ where x_z is a unique solution to $x' = f(x) - x$, $x(0) = z$. By Proposition II.3, z, $y \in D$ and $p_n(S(t)z - S(t)y) \leq \exp((\omega_n - 1)t)p_n(z - y)$ for $t \geq 0$ and all n. Since $S(t): D \to D$, the contraction mapping theorem applied for each $t > 0$, together with the fact that $S(t)$ commutes with $S(s)$ for t, $s \geq 0$, implies the existence of a unique $\omega \in D$ with $S(t)w = w$ for $t \geq 0$. This completes the proof. ■

We recall that $f: D \to X$ is a contraction if $p_n(f(x) - f(y)) \leq k_n p_n(x - y)$, for x, $y \in D$, $n \geq 1$, and some $0 \leq k_n < 1$.

Corollary II.5. *Let X be a Fréchet space, D a closed and locally bounded subset of X, and $f: D \to X$ a contraction. Suppose that* (CM) *is satisfied with f replaced by $f - I$. Then f has a unique fixed point in D.*

Let us point out that Theorem II.4 and Corollary II.5 are not true if (CM) is replaced by (C2), i.e. if f is only weakly inward on D. In a Banach space each weakly inward contraction has a unique fixed point. Using Theorem IV.2 of [1] we may obtain the following corollary.

Corollary II.6. *Let X be a Fréchet space and D a convex weakly compact subset of X. Assume that $f: D \to X$ is a contraction. If f is weakly inward, i.e. if $f - I$ satisfies* (C2), *then f has a unique fixed point in D.*

In this regard we note that when X is a reflexive Fréchet space then each closed convex and bounded subset of X is weakly compact.

Finally we want to state an analog of Corollary II.6 for f α-condensing. First we recall some definitions. If α is the Kuratowski measure of noncompactness (see [9]) we say that $f: D \to X$ is α-Lipschitz if $[\alpha(f(B))](p_n) \leq k_n[\alpha(B)](p_n)$ for all bounded $B \subset D$, $n \geq 1$ and for some $k_n \geq 0$. f is called α-condensing if $[\alpha(F(B))](p_n) < [\alpha(B)](p_n)$ whenever $[\alpha(B)](p_n) > 0$.

We need the following result.

Proposition II.7. *Let X be a metrizable* l.c.t.v.s. *and D a convex, weakly compact subset of X. In addition, let us assume that $f: D \to X$ is a continuous and α-Lipschitz map with bounded range. If f satisfied condition* (C2) *then* (I.1) *has a solution on* $[0, T]$ *for any* $T > 0$.

Proof. First we shall show that there exists a bounded abs. convex subset B of X with $D \cup f(D) \subset B$ such that on D the weak topology of X and the weak topology of X_B coincide. Therefore, D will be weakly compact in X_B. Indeed, let B be a bounded and abs. convex subset of X such that $D \cup f(D) \subset B$ and on D the topologies induced by X and X_B are the same. Clearly, the dual X^* of X is contained in X_B^*. Therefore, on D the weak topology of X_B is

stronger than the weak topology of X. Furthermore, for a convex subset D of the l.c.t.v.s. $Y = X_B$, the fact of weak compactness depends only on the topology induced by Y (original topology) on D. Hence D is weakly compact in X_B. Moreover, every coarser Hausdorff topology on D coincides with the weak topology of X_B on D, which demonstrates the assertion.

Take $x_n \in D$ and $h_n \to 0$ such that $z_n = (x_n - x)/h_n - f(x) \to 0$. Because $x_n \to x$, continuity of f implies that $y_n = (x_n - x)/h_n - f(x_n) = z_n + f(x) - f(x_n) \to 0$. Let V be an abs. convex nbhd. of zero in X and $\varepsilon > 0$. For each $x \in D$, there is an abs. convex and bounded subset B_x of X such that $y_n \to 0$ in X_{B_x}. By increasing B_x, if necessary, we can assume that $B \subset B_x$. Next take $\delta_x > 0$ such that $\delta_x B_x \subset V$. Then there exists $m \geq 1$ and $h_m \leq \varepsilon$ with $y_m \in \delta_x B_x$. Now if V_w is a weakly open nbhd. of zero in X_B then $x \in h_m f(x_m) - x_m + h_m \delta_x (B_x + V_\omega) \cap B_x = N_x$ and N_x is a weakly open nbhd. of x in (D, weak topology of X_B). By the above considerations, D is weakly compact in X_B, and therefore $D \subset \bigcup_{i=1}^n N_{x_i}$ for some $n \geq 1$. For $B_0 = \bigcup_{i=1}^n \delta_{x_i} B_{x_i}$ and each $x \in D$ we have $(x_h - x)/h - f(x_h) \in B_0 \cap V$ for some $0 < h \leq \varepsilon$ and $x_h \in D$. In other words, we have shown that (C1) is satisfied with $f(x)$ replaced by $f(x_h)$ in the last inclusion.

Next, D, as a weakly compact set, is a complete subset of X. Therefore, using the same ideas as in the proof of Proposition II.1, we can obtain a (V, ε)-approximate solution to (I.1) which is a backward approximate solution as compared to a forward approximate solution obtained previously. Let $x_n(t)$ be a (V_n, ε_n)-approximate solution, where $V_{n+1} \subset V_n$ is a countable base of nbhds. of zero in X and $\varepsilon_n \downarrow 0^+$. For $k \geq 1$ and $n \geq 1$ we define $m_n(t) = [\alpha(\Omega_k(t))](p_n)$ with $\Omega_k(t) = \{x_i(t) \colon i \geq k\}$. From the fact that f is α-Lipschitz it follows that for each $\varepsilon > 0$ and $n \geq 1$ there exists $k_n \geq 1$ such that

$$D_- m_n(t) \overset{\text{def}}{=} \liminf_{h \to 0^+} \frac{1}{h}[m_n(t) - m_n(t-h)] \leq k_n m_n(t) + 2\varepsilon$$

for $t \in [0, T]$. Next, standard arguments (see [10] Lemma 1.6.1) show that for each $n \geq 1$ and $t \in [0, T]$, $[\alpha(\{x_n(t)\})](p_n) = 0$. Finally, since D is a complete subset of X, the set $cl\{x_n(t)\}$ is compact for $t \in [0, T]$. Corollary II.2 completes the proof. ■

Theorem II.8. *Let X be a metrizable locally convex topological vector space. In addition, let us assume that*

(i) *D is a convex and weakly compact subset of X.*
(ii) *$f \colon D \to X$ is a continuous and α-condensing map with bounded range.*
(iii) *f is weakly inward.*

Then f has a fixed point in D.

Proof. Proposition II.7 shows that the initial value problem $x' = f(x) - x$, $x(0) = z \in D$, has a solution in D. This implies that condition (C1) for $f - I$ is satisfied with B independent of V. Next, since X is metrizable, $f - I$ satisfies (C1) in some X_B with bounded B containing $D \cup f(D)$. Now,

using similar ideas as in the proof of Theorem 1 of [11] we show that there exists a convex and compact $D^* \subset D$ such that $f - I$ satisfies (C2) on D^*, or equivalently, f is weakly inward on D^*. Finally, Theorem 4.1 of [12] implies that f has a fixed point in $D^* \subset D$. ■

Theorem II.8 and Proposition II.7 are also true if the assumption that f is α-condensing is replaced by the condition that f is b-condensing (see [13]). Theorem II.8 extends Reich's result ([13], Theorem 3.3). He assumed that f is inward on D. Furthermore, in [14] Reich conjectured that for every continuous weakly inward α-contraction $f: D \to X$, where D is a bounded closed convex subset of a Fréchet space X, the initial value problem $x' = f(x)$, $x(0) = x_0 \in D$ has a solution on $[0, T]$, $T > 0$. The example given in [1] shows that this is not true in general.

Corollary II.9. *Let D be a convex and weakly compact subset of metrizable topological vector space X. In addition, let us assume that $f: D \to X$ is a continuous and α-Lipschitz map with bounded range. Then the conditions* (C1), (C2), *and* (CM) *are equivalent.*

References

1. J. Polewczak, *Ordinary differential equations on closed subsets of locally convex spaces with applications to fixed point theorems*, (to appear in J. Math. Anal. App.)
2. R. S. Phillips, *Integration in convex linear topological spaces*, Trans. Amer. Math. Soc. **47** (1940), 114.
3. E. Dubinsky, *Differential equations and differential calculus in Montel spaces*, Trans. Amer. Math. Soc. **110** (1964), 1.
4. V. M. Millionscikov, *A contribution to the theory of differential equations $dx/dt = f(x, t)$ in locally convex spaces*, Soviet Math. Doklady **1** (1960), 228.
5. T. Yuasa, *Differential equations in a locally convex space via the measure of noncompactness*, J. Math. Anal. Appl. **84** (1981), 534.
6. R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982) 65.
7. M. DeWilde, *Closed graph theorems and webbed spaces*, Pitmann, London, 1978.
8. R. H. Martin, *Nonlinear operators and differential equations in Banach spaces*, J. Wiley and Sons, New York, 1976.
9. B. N. Sadovskii, *Limit-compact and condensing operators*, Bull. Russian Math. Surveys **27** (1982), 85.
10. V. Lakshmikantham and S. Leela, *Nonlinear differential equations in abstract spaces*, Pergamon Press, Oxford, 1981.
11. K. Deimling, *Fixed points of condensing maps*, in Volterra Equations, Lecture Notes in Math. 737, Springer Verlag, Berlin, 1979.
12. B. R. Halpern and G. M. Bergman, *A fixed-point theorem for inward and outward maps*, Trans. Amer. Math. Soc. **130** (1968), 353.
13. S. Reich, *Fixed points in locally convex spaces*, Math Z. **125** (1972), 17.
14. S. Reich, *A fixed point theorem for Fréchet spaces*, J. Math. Anal. Appl. **78** (1980), 33.

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