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THE IMAGE OF $H_*(BSO; Z_2)$ IN $H_*(BO; Z_2)$

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ABSTRACT. We construct explicit polynomial generators of the image of $H_*(BSO; Z_2)$ in $H_*(BO; Z_2)$.

Following the computation of the A-comodule algebra structure of $H_*(MSO; Z_2)$ by D. Pengelley [6], there has been renewed interest in studying bordism from the homology (as opposed to cohomology) point of view. Recently various families of generators for $H_*(BSO; Z_2)$ have been constructed by Bahri [1], Baker [2], Kochman [4, 5], and Pengelley [6, 7] using a variety of algebraic and geometric methods.

In this note we construct polynomial generators of the image of $H_*(BSO; Z_2)$ in $H_*(BO; Z_2)$ in terms of the canonical polynomial generators x_n, $n \geq 1$, of $H_*(BO; Z_2)$ (see [3, p. 479.]). These polynomial generators of $H_*(BSO; Z_2)$ have the distinctive property that they come from $H_*(BSO(3); Z_2)$. In that sense, they are simpler than the other known sets of generators.

In order to state our result we introduce the function τ which is defined on positive integers as follows: Let l be a positive integer and $l = 2^j + t$, $0 \leq t \leq 2^j$. We define $\tau(l) = t$ and and put $x_0 = 1$.

1. **Definition.** We define a sequence of elements $y_2, y_3, y_4, \dots, y_n, \dots$ of $H_*(BO; Z_2)$ as follows:

(a) If n is a power of 2, then $y_n = (x_{n/2})^2$.

(b) Let n be different than a power of 2 and $n = 2^k + m \cdot 2^{k+1}$, $0 \leq k$, $1 \leq m$. Then

$$y_n = x_n + x_{2^k} \cdot x_{m2^k}^2 + \sum_{2^k \leq a} \binom{a-1}{2^k - i - 1} \cdot x_i x_a x_b$$
$$+ \sum_{0<a<2^k} \binom{a-1}{\tau(2^k - i)} \cdot x_i x_a x_b,$$

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where the first summation extends over all ordered triples (i, a, b), with $i \leq a \leq b$, $i+a+b=n$, and $0 \leq i < 2^k$, and the second summation extends over the same range as the first, with the additional restriction $i + 2\tau(2^k - i) \leq a$.

Now we can describe our main result.

2. **Theorem.** *The image of* $H_*(BSO; Z_2)$ *in* $H_*(BP; Z_2)$ *is a polynomial algebra with polynomial generators* $y_2, y_3, \dots, y_n, \dots$.

Conjecturing the result was the main task. That took a lot of experimentation. Once the right conjecture was made, the proof was routine.

Before giving the proof we describe some of the generators.

If $n = 2m+1$, $m \geq 1$, then

$$y_n = x_n + x_1 x_{2m} + x_2 x_{2m-1} + \cdots + x_m x_{m+1} + x_1 x_m^2.$$

If $n = 4m+2$, $m \geq 1$, then

$$\begin{aligned} y_n = x_n &+ x_1 x_{4m+1} + (x_2 x_{4m} + x_4 x_{4m-2} + \cdots + x_{2m} x_{x2+2}) \\ &+ (x_1 x_1 x_{4m} + x_1 x_2 x_{4m-1} + \cdots + x_1 x_{2m} x_{2m+1}) + x_2 x_{2m}^2. \end{aligned}$$

If $n = 8m+4$, $m \geq 1$, then

$$\begin{aligned} y_n = x_n &+ x_1 x_{8m+3} + x_2 x_{8m+2} + x_3 x_{8m+1} \\ &+ \sum_{j=1}^{k} x_{4j} x_{8m+4-4j} + \sum_{j=2}^{[(n-2)/4]} x_2 x_{2j} x_{n-2j-2} \\ &+ \sum_{j=3}^{[(n-3)/2]} x_3 x_j x_{n-j-3} + x_4 x_{4m}^2 \\ &+ \sum x_1 x_j x_{n-j-1} + x_2 x_2 x_{8m} + x_2 x_3 x_{8m-1}, \end{aligned}$$

where the last summation extends over all integers j so that $4 \leq j \leq (n-2)/2$ and $j \equiv 0$ or $4 \bmod 4$.

In order to prove our main result we will need the following well-known lemma.

3. **Lemma.** *The image of* $H_*(BSO; Z_2)$ *in* $H_*(BO; Z_2)$ *is the kernel of the map*

$$d\colon H_*(BO; Z_2) \to H_*(BO; Z_2)$$

which is a derivation such that $d(x_n) = x_{n-1}$ *for* $n = 1, 2, 3, \dots$.

Proof. See [3, Proposition 4.2].

Proof of Theorem 2. It is easy to observe that the elements of $H_*(BO; Z_2)$, $y_2, y_3, y_4, \dots, y_n, \dots$, are algebraically independent. So, it is enough to prove that the y_n's belong to the kernel of d. The case where n is a power of 2 is obvious, since d is a derivation. Let us turn our attention to the other case. Using the definition of d, described in the previous lemma, one can calculate directly that $d(y_n) = 0$, where n is not a power of 2. The proof is direct. It consists of considering many cases, and using the identities

$$\binom{s+1}{r+1} = \binom{s}{r} + \binom{s}{r+1}$$

and $\tau(l+1) = \tau(l) + 1$ if $(l+1)$ is not a power of 2.

Let us give an idea of what the details look like. Let $2^k < a < b$, $0 < i$, $i + a + b = n$. Clearly

$$d(x_i x_a x_b) = x_{i-1} x_a x_b + x_i x_{a-1} x_b + x_i x_a x_{b-1}.$$

Let us examine the coefficient of, let us say, $x_{i-1} x_a x_b$ in the expansion of $d(y_n)$. The term $x_{i-1} x_a x_b$ will appear by the derivation of $x_i x_a x_b$, $x_{i-1} x_{a+1} x_b$, and $x_{i-1} x_a x_{b+1}$ with respective coefficients

$$\binom{a-1}{2^k - i - 1}, \quad \binom{a}{2^k - i}, \quad \binom{a-1}{2^k - i}$$

whose sum is zero mod 2. So $x_{i-1} x_a x_b$ will disappear from $d(y_n)$. Analogously we check all the possibilities of i, a, b.

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