THE IMAGE OF $H_*(BSU; \mathbb{Z}_p)$ IN $H_*(BU; \mathbb{Z}_p)$

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(Communicated by Haynes R. Miller)

Abstract. In this note we construct explicit polynomial generators for the image of $H_*(BSU; \mathbb{Z}_p)$ inside $H_*(BU; \mathbb{Z}_p)$.

Introduction

Recently various families of generators of the image of $H_*(BSO; \mathbb{Z}_2)$ inside $H_*(BO; \mathbb{Z}_2)$, have been constructed by Bahari [1], Baker [2], Kochman [4, 5], Papastavridis [7] and Pengelley [8, 9]. In this note we find relatively simple polynomial generators for the image of $H_*(BSU; \mathbb{Z}_p)$ inside $H_*(BU; \mathbb{Z}_p)$.

It is well known that

$$H_*(BU; \mathbb{Z}_p) = \mathbb{Z}_p[x_1, x_2, \ldots, x_n, \ldots],$$

where $p$ is a prime number and $x_1, x_2, \ldots, x_n, \ldots$ are the well-known elements of $H_*(BU; \mathbb{Z}_p)$, (see, for example, [2, Proposition 10]).

If $n$ is not a power of $p$, then nonnegative integers $a, b, r$, such that $n = apr + bpr$ and $0 < b < p$, are uniquely defined. We put $t = p^r$. Then we define

$$y_n = \sum_{i=0}^{n-2r} \sum_{j=0}^{n-2i} (-1)^{i+j} \binom{n-2i-j}{t-i} \binom{n-t-1-i-j}{t-1-i} x_i x_j x_{n-i-j}.$$

Remark. The symbol $\binom{n}{k}$ is the binomial coefficient. By definition we put

$$a \binom{b}{-1} = 1 \quad \text{for } a \geq 0 \text{ and } b \geq -1.$$

If $n = p^r$ and $r > 0$, then we define $y_n = (x_{n/p})^p$.

In this note, we will prove the following theorem.

Theorem 1. The image of $H_*(BSU; \mathbb{Z}_p)$ in $H_*(BU; \mathbb{Z}_p)$, under the obvious monomorphism, is the polynomial algebra generated by $y_2, y_3, \ldots, y_n, \ldots$. 
Proof of Theorem 1. From B. Gray's paper, [3] we will need the following lemma.

Lemma 2. Let \( d : H_*(BU; \mathbb{Z}_p) \to H_*(BU; \mathbb{Z}_p) \) be the derivation defined by \( d(x_n) = x_{n-1} \). Then the sequence

\[
0 \to H_*(BSU; \mathbb{Z}_p) \to H_*(BU; \mathbb{Z}_p) \xrightarrow{d} H_*(BU; \mathbb{Z}_p) \to 0
\]

is exact.

Proof. See [1, Proposition 4.2]. The proof refers to \( BO \), but it works the same for \( BU \).

It is easy to observe, that if \( n \) is not a power of \( p \), then \( y_n \) is indecomposable in \( H_*(BU; \mathbb{Z}_p) \). The reason is the following: Let \( n = ap^{r+1} + bp^r \), with \( r \geq 0 \) and \( 0 < b < p \). Then the coefficient of \( x_n \) in the expression for \( y_n \) is

\[
\frac{n}{t} \binom{n-t-1}{t-1} = (ap + b) \cdot \frac{(ap^{r+1} + (b-1)p^r - 1)}{p^r - 1} \equiv b \neq 0 \mod p.
\]

(We use the well-known formula which computes the binomial coefficient \( \mod p \)). From this remark it follows that the elements \( y_2, y_3, \ldots, y_n, \ldots \) are polynomially independent. Furthermore, it is obvious, that the graded polynomial algebra \( \mathbb{Z}_p[y_2, y_3, \ldots, y_n, \ldots] \) has in each degree the same \( \mathbb{Z}_p \)-dimension as \( H_*(BSU; \mathbb{Z}_p) \). So, Lemma 2 tells us that, in order to prove Theorem 1, it is enough to prove that the \( y_n \)'s belong to the kernel of the derivation \( d \). The case where \( n \) is a power of \( p \) presents no problem because in this case \( y_n \) is a \( p \)-th power. Our next lemma is all we need to settle the case where \( n \) is not a power of \( p \).

Lemma 3. Let us define

\[
c_{i,j} = \begin{cases} 1, & \text{if } i = 0 \text{ and } j \geq -1. \\ \frac{1}{2} \binom{i}{j} + \binom{i+1}{j+1}, & \text{if } i \geq 1 \text{ and } j \geq i - 1.
\end{cases}
\]

Then \( c_{i+1,j+1} - c_{i,j} - c_{i+1,j} = 0 \) for \( i \geq 0 \) and \( j \geq i - 1 \).

Proof. It is obvious. Just direct calculations.

Now, with this lemma in our hand, it is an easy matter to prove that \( y_n \) belongs to the kernel of the derivation \( d \) for the case where \( n \) is not a power of \( p \). We calculate \( d(y_n) \), using the usual rule of the derivation of a product, and our last lemma is all we need to see that we get zero.

Comment. The referee and the editor, H. Miller, suggested a comment on the discovery of the formula for \( y_n \). Each \( y_n \) is a linear combination of monomials of length \( \leq 3 \). The problem is to guess the coefficients of this linear combination. The double summation covers a rectangle in the \( i - j \) plane. The edges of the rectangle ensure the appearance of the indecomposable monomial \( x_n = x_0x_0x_n \). The coefficients are combinations of binomial coefficients designed to produce cancellations in \( d(y_n) \), according to the Pascal triangle.
Bibliography


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