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SHORT PROOFS OF TWO HYPERGEOMETRIC SUMMATION FORMULAS OF KARLSSON

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ABSTRACT. Karlsson [2] gave elegant proofs of two hypergeometric summation formulas conjectured by Gosper, that were mentioned in [1]. Here I give new proofs that are much shorter, but less elegant.

Theorem 1 ([1, formula (6.5)]).

$$\sum_k \frac{n!(n-1/4)!}{(n-k)!(n-k-1/4)!(2n+k+1/4)!k!9^k} = \frac{(-1/3)!(1/12)!2^{6n}}{(1/4)!(n-1/3)!(n+1/12)!3^{5n}}. \tag{1}$$

Proof. Let $R(n)$ and $F(n,k)$ be the sum and the summand respectively on the left. Since the theorem is obviously true for $n=0$, it would follow by induction once we show that

$$27(12n+13)(3n+2)R(n+1)-(256)R(n)=0. \tag{2}$$

Let

$$G(n,k) := -16\frac{52n^2+91n+18+16kn-40k-32k^2}{(8n+4k+5)(8n+4k+9)}F(n,k).$$

It is readily verified that

$$27(12n+13)(3n+2)F(n+1,k)-(256)F(n,k) = G(n,k)-G(n,k-1), \tag{3}$$

and we get (2) by summing (3) with respect to k. □

Theorem 1$'$ ([1, formula (6.6)]).

$$\sum_k \frac{n!(n-1/4)!}{(n-k)!(n-k-1/4)!(2n+k+5/4)!k!9^k} = \frac{(1/3)!(5/12)!2^{6n}}{(5/4)!(n+1/3)!(n+5/12)!3^{5n}}. \tag{1$'$}$$

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Proof. Let $R(n)$ and $F(n,k)$ be the sum and the summand respectively on the left. Since the theorem is obviously true for $n = 0$, it would follow by induction once we show that

$$(2') \qquad 27(12n+17)(3n+4)R(n+1) - (256)R(n) = 0.$$

Let

$$G(n,k) := -16\frac{52n^2 + 143n + 36 + 16kn - 68k - 32k^2}{(8n+4k+9)(8n+4k+13)}F(n,k).$$

It is readily verified that

$$(3') \quad 27(12n+17)(3n+4)F(n+1,k) - (256)F(n,k) = G(n,k) - G(n,k-1),$$

and we get $(2')$ by summing $(3')$ with respect to k. □

References

1. I. M. Gessel and D. Stanton, *Strange evaluations of hypergeometric series*, SIAM J. Math. Anal. **13** (1982), 295–308.
2. P. W. Karlsson, *On two hypergeometric summation formulas conjectured by Gosper*, Simon Stevin, J. Pure and Appl. Math. **60** (1986), 286–296.

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