TWO CLASSES OF FRÉCHET-URYSOHN SPACES

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Abstract. Arhangel'skii introduced five classes of spaces, \( \alpha_i \)-spaces \((i < 5)\), which are important in the study of products of Fréchet-Urysohn spaces. For each \( i < 5 \), each \( \alpha_i \)-space is an \( \alpha_{i+1} \)-space and it follows from the continuum hypothesis that there are countable \( \alpha_{i+1} \)-spaces which are not \( \alpha_i \)-spaces. A \( \nu \)-space (\( \omega \)-space) is a Fréchet-Urysohn \( \alpha_1 \)-space (\( \alpha_2 \)-space). We show that there is a model of set theory in which each \( \alpha_2 \)-space (\( \omega \)-space) is an \( \alpha_1 \)-space (\( \nu \)-space).

Introduction

Arhangel'skii defines a point \( x \in X \) to be an \( \alpha_1 \)-point (\( \alpha_2 \)-point) if whenever \( F_n \) is a sequence converging to \( x \), for each \( n < \omega \), there is a sequence \( F \) converging to \( x \) such that \( F_n - F \) is finite (\( F_n \cap F \) is infinite) for each \( n < \omega \). Furthermore a point is called an \( \alpha_0 \)-point if it has a countable neighbourhood base. A space is called an \( \alpha_1 \)-space if each point is an \( \alpha_1 \)-point. A space is Fréchet-Urysohn if whenever a point is in the closure of a set there is a sequence from it converging to the point.

Nyikos has shown that there is a countable \( \omega \)-space which is not first countable [Ny1, Ny2]. In [Ny2] Nyikos asks if there is a countable \( \omega \)-space which is not a \( \nu \)-space and a countable \( \nu \)-space which is not first-countable. Nyikos [Ny2] produces examples of countable \( \omega \)-spaces which are not \( \nu \)-spaces from a special set-theoretic assumption following from, for example, Martin's Axiom (even \( b < d \)). Nogura [N] has shown that MA (even \( b = d \)) implies there is an example of a countable \( \nu \)-space which is not first countable. It is shown in [DS] that it is consistent that each countable \( \nu \)-space is first-countable.

Gruenhage [G] introduced \( \nu \)-spaces and Sharma [Sh] obtained their characterization in terms of \( \alpha_2 \)-spaces. The term \( \nu \)-space seems to be due to Nyikos.

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\(\omega\)-SPLITTING FAMILIES FROM \(\alpha_2\)-POINTS

1. Definition. We say that \(\mathcal{X} \subseteq [\omega]^\omega\) is \(\omega\)-splitting if for each countable family \(\{A_n : n \in \omega\} \subseteq [\omega]^\omega\) there is an \(X \in \mathcal{X}\) such that \(|A_n \cap X| = |A_n - X|\) for each \(n \in \omega\). We shall say that \(X\) is \(M\)-splitting if \(|A \cap X| = |A - X|\) for each \(A \in M \cap [\omega]^\omega\).

Let \(x\) be a point in a space \(Y\) and suppose \(F_n \in [Y]^\omega\) is a sequence converging to \(x\) for each \(n \in \omega\). Identify \(\bigcup F_n\) with \(\omega\) and define \(\mathcal{X}\) to be the set of \(X\) which “think” that \(x\) is an \(\alpha_1\)-point. That is, define \(\mathcal{X}(x, \langle F_n \rangle) = \{X \subseteq [\omega] : \exists F \subseteq Y. (F\text{ converges to }x \text{ and } (X \cap F_n) - F\text{ is finite for each }n \in \omega)\}\). The definition of \(\mathcal{X}(x, \langle F_n \rangle)\) of course depends on the identification of \(\bigcup F_n\) with \(\omega\), but only up to a permutation on \(\omega\) and this will never matter to us.

2. Lemma. If \(F_n \in [Y]^\omega\) \((n \in \omega)\) converges to an \(\alpha_2\)-point \(x\), then \(\mathcal{X} = \mathcal{X}(x, \langle F_n \rangle)\) is an \(\omega\)-splitting family.

Proof. Assume that \(\bigcup F_n\) is identified with \(\omega\) and let \(\{A_k : k \in \omega\} \subseteq [\omega]^\omega\). Let \(I = \{j \in \omega : A_j \cap F_n\text{ is finite for each }n\}\). Choose for each \(n\), a finite subset \(H_n\) of \(F_n \cup \bigcup_{m < n} F_m\) so that \(A_j - H\text{ is finite for each }j \in I\text{ where }H = \bigcup H_n\). Now for each \(j \in \omega - I\), choose \(n_j \in \omega\) so that \(A_j \cap F_{n_j}\) is infinite. Now choose an infinite \(B_j \subseteq A_j \cap F_{n_j}\) so that \(B_j \cap B_k = \emptyset\) for \(j \neq k\) in \(\omega - I\). Now since \(x\) is an \(\alpha_2\)-point, there is a sequence \(F\) converging to \(x\) such that \(F \cap B_j\) is infinite for each \(j \in \omega - I\). It follows that \(F \cup H\) hits each \(A_j\) in an infinite set and that \(F \cup H \in \mathcal{X}\). Finally any infinite subset of \(F \cup H\) is a member of \(\mathcal{X}\) hence there is an \(X \in \mathcal{X}\) which splits \(\{A_k : k \in \omega\}\).

3. Lemma. If every \(\omega\)-splitting family contains an \(\omega\)-splitting family of cardinality less than \(b\), then each \(\alpha_2\)-point is an \(\alpha_1\)-point.

Proof. Let \(x\) be an \(\alpha_2\)-point of a space \(Y\) and assume \(F_n\) is a sequence converging to \(x\) for each \(n \in \omega\). Since we wish to show that \(x\) is an \(\alpha_1\)-point we may assume that the \(F_n\)'s are pairwise disjoint and that \(\bigcup F_n = \omega\). Let \(\mathcal{X} = \mathcal{X}(x, \langle F_n \rangle)\) be defined as above. By Lemma 2 and the hypothesis of this lemma, there is an \(\omega\)-splitting family \(\mathcal{X}' \subseteq [\mathcal{X}]^{<b}\). For each \(A \in \mathcal{X}'\), choose \((\text{by the definition of }\mathcal{X})\) a sequence \(F_A\) converging to \(x\) and \(f_A \in [\omega]^{<\omega}\) so that \((A \cap F_n) - f_A \subseteq f_A(n)\) for each \(n \in \omega\). Now choose \(f \in [\omega]^{<\omega}\) so that \(f_A <^* f\) for each \(A \in \mathcal{X}'\) which we may do since \(|\mathcal{X}'| < b\). We claim that \(F = \bigcup F_n - f(n)\) converges to \(x\), which would show that \(x\) is an \(\alpha_1\)-point. Indeed, assume \(F\) does not converge to \(x\) and choose \(F' \in [F]^\omega\) such that \(x\) is not a limit point of \(F'\). Since \(\mathcal{X}'\) is splitting, choose \(A \in \mathcal{X}'\) so that \(A \cap F'\) is infinite. However this contradicts that \(F_A\) converges to \(x\) since \(A \cap F' - F_A\) is finite.
4. **Remark.** We could replace the notion of \( \omega \)-splitting in Lemma 3 by what one might call “\( \omega \)-hitting”.

\( \omega \)-splitting families and Laver forcing

5. **Theorem.** In a model obtained by adding \( \omega_2 \)-Laver reals to a model of CH, every \( \omega \)-splitting family contains an \( \omega \)-splitting family of cardinality \( \omega_1 \). Hence in this model each \( \alpha_2 \)-space (\( w \)-space) is an \( \alpha_1 \)-space (\( v \)-space).

The theorem follows from the following four results. Proposition 6 is a collection of standard facts about Laver forcing, Lemma 7 is a standard reflection argument, Lemma 8 is a special case of a general preservation scheme proven in [S2] and Lemma 9 is new. Recall that \( T \subseteq \text{L} \) (the Laver poset defined in [L]) if \( T \subseteq \text{cf} \omega \) has a root \( t_0 = \text{root}(T) \) and for \( t_0 \leq t \in T \) \( \{ n : \text{f}^t n \in T \} \) is infinite. \( \text{L} \) is ordered by inclusion. For \( \lambda \leq \omega_2 \) let \( P_\lambda \) be the countable support \( \lambda \)-stage iteration of the forcing notion \( \text{L} \).

6. **Proposition.** [CH] \( P_{\omega_2} \) is an \( \omega_2 \)-cc proper poset such that \( 1 \Vdash_p \ b = c = |\lambda \cdot \omega_1|^\lambda \). Furthermore if \( \mu < \lambda \) then \( P_\lambda \) is forcing isomorphic to \( P_\mu \cdot P_\lambda \).

For proofs of the various assertions in Proposition 6 we refer the reader to [L] and [S1].

7. **Lemma.** [CH] Let \( \{ X_\alpha : \alpha < \omega_2 \} \) be \( P_{\omega_2} \)-names such that \( 1 \Vdash \{ X_\alpha : \alpha < \omega_2 \} \subseteq [\omega]^{\omega} \) is \( \omega \)-splitting, then there is a \( \lambda < \omega_2 \) such that \( 1 \Vdash \{ X_\alpha : \alpha < \lambda \} \) is \( \omega \)-splitting.

**Notation.** If \( p \) is a member of a poset \( P \) and \( M \) is a set, “\( p \Vdash \ X \) is \( M \)-splitting” will abbreviate \( p \Vdash \ |A \cap X| = |A - X| \) for each \( P \)-name \( A \in M \) such that \( p \Vdash \ A \in [\omega]^{\omega} \).

We shall say that a poset \( P \) is \( \omega \)-splitting if the following are satisfied: whenever \( P \in M \), where \( M \) is a countable elementary submodel of \( H(\theta) \) for any sufficiently large \( \theta \), \( p \in M \cap P \) and \( X \in M \)-splitting then there is some \( q < p \) which is \( (M,P) \)-generic and such that \( q \Vdash X \) is \( M \)-splitting.

Note that an iteration of finitely many \( \omega \)-splitting posets is again \( \omega \)-splitting; hence proper.

8. **Lemma.** If \( P_\delta = \langle (P_\alpha, Q_n) : \alpha < \delta \rangle \in M \) is a countable support iteration of \( \omega \)-splitting (hence proper) posets then \( P_\delta \) is also \( \omega \)-splitting.

9. **Lemma.** Let \( M \) be a countable elementary submodel of \( H(\omega_2) \) and let \( T \subseteq \text{L} \cap M \), then if \( X \) is \( M \)-splitting there is an \( (M,L) \)-generic \( T' < T \) such that \( T' \Vdash X \) is \( M \)-splitting. Therefore \( \text{L} \) is \( \omega \)-splitting.

It may be worthwhile to record the following corollary to the above results.

10. **Corollary.** If a family is \( \omega \)-splitting then it will still be \( \omega \)-splitting after forcing with the countable support iteration of Laver forcing. Furthermore in any
model obtained by adding iteratively \( \omega_2 \)-Laver reals the splitting number, \( \mathfrak{s} \), will be \( \omega_1 \).

Before proving Lemmas 7–9 let us indicate how Theorem 5 now follows. Let \( G \) be \( \mathbf{P}_{\omega_2} \)-generic over \( V \) (a model of CH). Let \( \mathcal{X} = \{ X_\alpha : \alpha < \omega_2 = \mathfrak{c} \} \) be an \( \omega \)-splitting family. Choose \( \mathbf{P}_{\omega_2} \)-names \( \{ X_\alpha : \alpha < \omega_2 \} \in V \) so that \( 1 \Vdash \mathcal{X} = \{ X_\alpha : \alpha < \omega_2 \} \). By Lemma 7, there is a \( \lambda < \omega_2 \) so that \( 1 \Vdash_{\mathbf{P}_\lambda} \{ X_\alpha : \alpha < \lambda \} \) is \( \omega \)-splitting. Let \( G_\lambda = G \cap \mathbf{P}_\lambda \); hence \( V[G_\lambda] \models \{ X_\alpha : \alpha < \lambda \} \) is \( \omega \)-splitting. Since \( \mathbf{P}_{\omega_2} \cong \mathbf{P}_\lambda \ast \mathbf{P}_{\omega_2} \), \( V[G] \models \{ X_\alpha : \alpha < \lambda \} \) is \( \omega \)-splitting by Lemmas 8 and 9.

It remains to prove the Lemmas.

**Proof of Lemma 7.** By Proposition 6, we may assume that for each \( \alpha < \omega_2 \), there is an \( f(\alpha) < \omega_2 \) such that \( X_\alpha \) is a \( \mathbf{P}_{f(\alpha)} \)-name. Also if \( G \) is \( \mathbf{P}_{\omega_2} \)-generic, \( V[G] \models \) for each \( \alpha < \omega_2 \), there is a \( g(\alpha) < \omega_2 \) such that for each set \( M \in [\omega_2]^{\omega_2} \cap V[G_\alpha] \) there is an \( X \in \{ \text{val}(X_\alpha^C, G) : \beta < g(\alpha) \} \) which is \( M \)-splitting (since \( V[G_\alpha] \models \mathfrak{c} = \omega_1 \) ). Since \( \mathbf{P}_{\omega_2} \) is \( \omega_2 \)-cc we may assume that \( g \in V \). Now let \( h \) be a continuous strictly increasing function from \( \omega_2 \) into \( \omega_2 \) such that \( f(\alpha) + g(\alpha) < h(\alpha + 1) \) for all \( \alpha < \omega_2 \). Choose \( \lambda < \omega_2 \) such that \( h(\lambda) = \lambda \); it follows that \( 1 \Vdash_{\mathbf{P}_\lambda} \{ X_\alpha : \alpha < \lambda \} \) is \( \omega \)-splitting.

**Proof of Lemma 8.** Technically we make the inductive assumption that for each \( \beta < \alpha < \delta \) and each \( p \in \mathbf{P}_\beta \) we have that \( p \Vdash -_{\mathbf{P}_\alpha} P_{\alpha} / P_\beta \) is \( \omega \)-splitting; where, as usual, \( P_\alpha / P_\beta \) denotes the \( \mathbf{P}_\beta \)-name satisfying the equation \( P_\alpha = \mathbf{P}_\beta \ast (P_\alpha / P_\beta) \). However in proving the inductive step we can just force with \( \mathbf{P}_\beta \) and work in the extension. Therefore we may make the above inductive assumption and complete the proof by showing that \( \mathbf{P}_\delta \) is \( \omega \)-splitting. As remarked when we defined the notion of an \( \omega \)-splitting poset the proof is trivial in case \( \delta \) is a successor ordinal. Now let \( \theta \) be a large enough cardinal (i.e. \( |\mathcal{P}(\mathbf{P}_\delta)| < \theta \) and let \( \mathbf{P}_\delta \in M \) where \( M \) is a countable elementary submodel of \( H(\theta) \). Suppose further that \( p \in M \cap \mathbf{P}_\delta \) and that \( X \) is \( M \)-splitting. Let \( \{ A_n : n \in \omega \} \) index the set of \( \mathbf{P}_\alpha \)-names of subsets of \( \omega \) which are in \( M \). We may as well assume that \( cf(\delta) = \omega \) since we will be choosing \( q \) to be \( M \)-generic and this \( q \) will force that the \( A_n \)'s are essentially \( P_{M \cap \delta} \)-names. We may therefore choose a countable increasing sequence of ordinals cofinal in \( \delta \) and by our inductive assumption we may as well assume that \( \delta = \omega \).

As in [S2], we shall choose sequences \( p_n, q_n, k_n, m_n \), by induction on \( n \), so that:

1. \( p_n \in \mathbf{P}_\omega \), \( q_n \in \mathbf{P}_n \), and \( k_n \), \( m_n \) are \( \mathbf{P}_\omega \)-names of integers,
2. \( p_{n+1} \models n = p_n \land p_{n+1} < p_n \), \( q_{n+1} \models n = q_n \) and \( q_n < p_n \),
3. for \( n > 0 \), \( q_n \land p_n \models A_n \) is finite or \( m_n \in X \cap A_n \) and \( k_n \in A_n \setminus X \),
   \( q_n \land p_n \) is meant to denote the element \( (\langle q_n \land p_n \rangle n) \land p([n, \omega) \in P_\omega) \)
4. \( q_n \) is \( \mathbf{P}_M \)-generic and \( q_n \models -_{\mathbf{P}_n} X \) is \( M \)-splitting, and
5. \( q_n \models -_{\mathbf{P}_n} X \in M \).

Before we begin the induction, let us comment on what (5) means. It is not the case that \( p_n \) will be in \( M \) but \( q_n \models - (\exists p' \in M) \) such that \( p' / P_n = p_n / P_n \).
where \( p/P_n \) denotes the \( P_\omega/P_n \)-name corresponding to \( p \) in the extension by \( P_n \). It is not even the case that we can bring the "\( \exists \)" sign outside the forcing statement. However condition 5 is essential in order for the induction to continue. Let \( p_1 = p \) where \( p \in P_\omega \cap M \) is as chosen above. Since \( p_1 \) is proper (and we have chosen \( \theta \) large enough) we may choose \( q_1 < p_1 \) to be \( (M, P_\nu) \)-generic such that \( q_1 \parallel \neg X \) is \( M \)-splitting. Suppose that \( p_n \) and \( q_n \) have been chosen. Let \( G_n \) be \( P_n \)-generic such that \( q_n \) and \( p_n \) are in \( G_n \). In \( V[G_n] \), let \( p'_n \) be the element of \( M \) so that \( p'_nP_n = p_nP_n \), and let \( B_{n+1} = \{ m : (\exists p \in P_\omega/P_n) \ p < p'_n/P_n \text{ and } p \parallel m \in A_{n+1} \} \). Here we are assuming that \( A_n/P_n \) is the \( P_\omega/P_n \)-name which results from evaluating the \( P_\omega \)-name \( A_n \) and similarly for \( p'_n \). Now \( M[G_n] \) is an elementary submodel of \( H(\theta)^{V[G_n]} \) which is a model of \( ZF-P \) (see [S1]) and \( p'_n \in M \), hence \( B_{n+1} \subseteq M[G_n] \). Since \( X \) is \( M[G_n] \)-splitting we may choose \( m_{n+1} \in X \cap B_{n+1} \) and \( p'_n/P_n < p'_n/P_n \), \( p' \in M \) (by elementarity) such that \( p'_n \parallel \neg m_{n+1} \in A_{n+1} \). Similarly we may choose \( p'_{n+1}/P_n < p'_{n+1}/P_n \) and \( k_{n+1} \notin X \) such that \( p'_{n+1} \in M \), \( p'_{n+1} \parallel \neg k_{n+1} \in A_{n+1} \). By assumption \( Q_n \) is \( \omega \)-splitting (in \( V[G_n] \)) hence we may choose \( q_{n+1} < p'_{n+1}(n) \), \( q_{n+1} \in Q_n \) so that \( q_{n+1} \) is \( (M[G_n], Q_n \)-generic and so that \( q_{n+1} \parallel \neg X \) is \( M[G_n] \)-splitting. Now we use the maximality principle to choose \( q_{n+1}, p_{n+1}, p_{n+1}' \) and the names \( m_{n+1} \) and \( k_{n+1} \) so as to satisfy (1)-(5). That is, \( q_n \parallel \neg X \) if \( X \) is \( M[G_n] \)-splitting and \( M[G_n] \) is an elementary submodel of \( H(\theta)^{V[G_n]} \) then there are \( p_{n+1}/P_n \in M[G_n], k_{n+1} \) and \( m_{n+1} \) as above. So we may choose a \( P_n \)-name, say \( p_{n+1}'' \), of an element of \( P_\omega/P_n \) and \( P_\omega \)-names \( k_{n+1} \) and \( m_{n+1} \) so that \( p_n \parallel \neg X \) if \( X \) is \( M[G_n] \)-splitting and \( M[G_n] \) is an elementary submodel of \( H(\theta)^{V[G_n]} \) then \( p_{n+1}'' \in M[G_n] \) and \( p_{n+1}'' \parallel \neg k_{n+1} \) and \( m_{n+1} \) are as above. Next we choose a \( P_n \)-name \( q_{n+1} \) for \( q_{n+1}'' \). We let \( p_{n+1}'' = p_n \wedge p_{n+1}' \) and \( q_{n+1} = q_n \wedge q_{n+1} \). It is clear that (1), (2) are satisfied. By [S1], \( q_{n+1} \) is \( (M, P_{n+1}) \)-generic and clearly \( q_{n+1} \parallel \neg X \) is \( M \)-splitting; hence (4) holds. The reason that (3), (5) are satisfied is that \( q_{n+1} \parallel \neg M[G_{n+1}] \) is an elementary submodel of \( H(\theta)^{V[G_{n+1}]} \), hence \( p_{n+1}'' \) has the desired properties.

Now if \( q \in P_\omega \) is such that \( q \parallel n = q_n \) for each \( n \in o \) then \( q \) is \( (M, P_\omega) \)-generic (see [S1]) and \( q < p_n \) for each \( n \in \omega \). It follows that \( q \parallel \neg X \) is \( M \)-splitting, since for each \( i < j < \omega \) there is an \( n < \omega \) such that \( \parallel A_j = A_i - j \) and so \( q \parallel n \wedge p_n \parallel \neg m \in X \cap A_i - j \) and \( k_n \in A_i - (X \cup j) \).

Proof of Lemma 9. Let \( T \subseteq L \cap M \) where \( M \) is a countable elementary submodel of \( H(\omega_3) \) and assume \( X \) is \( M \)-splitting. Fix indexings \( \{ A_n \} \) of \( M \cap \{ A \mid A \text{ is an } L \text{-name and } 1 \models A \in [\omega]^\omega \} \), and \( \{ D_n \} \) of \( M \cap \{ D \subseteq L \mid D \text{ is dense open} \} \). We shall inductively define a descending sequence \( \{ T_n \} \subseteq L \) (with \( T_0 = T \)) so that, for each \( n \in \omega \):

1. \( T_n \cap \omega = T_{n+1} \cap \omega \), and
2. If \( T' < T_n \) then there is a \( t \subseteq T' \) such that
   a. \( T_n(t) \in M \cap D_n \) and
(b) \((T_n)_t \models X \triangleleft A_n \neq \emptyset\) and \(A_n - X \neq \emptyset\).

If we accomplish this, then condition 1 guarantees that \(T' = \cap T_n \in L\). It is easy to see that condition 2(b) guarantees that \(T' \models X\) is \(M\)-splitting. Condition 2(a) actually has the double role of ensuring that \(T'\) is \(M\)-generic and allowing the induction to continue.

Following [L], if \(S', S \in L\) then we use \(S' <^0 S\) to denote the situation where \(S' < S\) and they have the same root. One of the key facts about Laver forcing from [L] is that if \(S \in L\) and \(\varphi\) is any sentence of the forcing language then there is an \(S' <^0 X\) such that either \(S' \models \varphi\) or \(S' \models \neg \varphi\). Therefore

\[
(*) = \begin{cases}
  \text{if } A \text{ is an } L\text{-name and } F \text{ is a finite set such that} \\
  S \models F \cap A \neq \emptyset, \text{ then there is an } S' <^0 S \text{ and an } x \in F \\
  \text{such that } S \models x \in A.
\end{cases}
\]

Let us assume that \(0 < n \in \omega\) and that \(T_{n-1}\) has been chosen as above. Let \(I\) be the set of members of \(T_{n-1} - \leq^\infty \omega\) which are minimal with respect to the property that \((T_{n-1})_t \in M\). Note that the minimality of the members of \(I\) and condition 2(a) guarantee that the collection \(\{(T_{n-1})_t | t \in I\}\) is an antichain in \(L\) which is maximal below \(T_{n-1}\). Now if we find, for each \(t \in I\), a condition \(T'_t <^0 (T_{n-1})_t\) satisfying condition 2, then we can define \(T_n\) to be \(\bigcup\{T'_t | t \in I\}\).

This works since \(\{T'_t | t \in I\}\) is a maximal-below-\(T_n\) antichain. For the same reason, repeated uses of Facts 1 to 3 finish the proof.

Fact 1. If \(S \in L \cap M\) and \(n \in \omega\), then there is an \(S' <^0 X\) such that the collection \(\{(S')_t | (S')_t \in D_n \cap M\}\) is predense below \(S'\).

Fact 2. If \(S \in L \cap M\) and \(n \in \omega\) there is an \(S' <^0 X\) such that the collection \(\{(S')_t | (S')_t \in M\text{ and } (S')_t \models X \cap A_n \neq \emptyset\}\) is predense below \(S'\).

Fact 3. If \(S \in L \cap M\) and \(n \in \omega\) there is an \(S' <^0 S\) such that the collection \(\{(S'_t)_t | (S'_t)_t \models A_n - X \neq \emptyset\}\) is predense below \(S'\).

Fact 1 is, of course, a well-known property of \(L\) and its proof is similar to the proof of Fact 2. In the proof of Fact 2 we are just using that \(X\) is \(M\)-splitting. Since \(\omega - X\) is also \(M\)-splitting, Fact 3 follows from Fact 2. Now let us prove Fact 2.

Let \(B_n = \{k \in \omega | (\exists S' <^0 S) S' \models k \in A_n\}\). Let us first suppose that \(B_n\) is infinite. In this case, \(B_n \in [\omega]^{\omega} \cap M\), hence we have that \(X \cap B_n \neq \emptyset\).

It follows that we may choose \(S' <^0 S, S' \in M\) and \(k \in X \cap B_n\) so that \(S' \models k \in A_n\)—which certainly suffices.

Now let us suppose that \(\max(B_n) < m\). Let \(I\) be the set of minimal elements of \(\{t \in S | (\exists m_t \geq m) (\exists S'_t <^0 X_t) S'_t \models m_t \in A_n\}\). Since \(1 \models (\exists k \geq m) k \in A_n\) and the members of \(I\) are minimal, \(\{S_t | t \in I\}\) is a maximal-below-\(S\) antichain of \(L\). For each \(t \in I\), fix a minimal \(m_t\) and an \(S'_t \in M\) as in the description of \(I\); hence \(\{m_t | t \in I\} \in M\). We shall show that \(S' = \bigcup(S'_t | t \in I\) and \(m_t \in X\) works. That is, we prove that \(S' \in L\), \(S' <^0 S\) and simply note that \(\{S'_t | t \in I\) and \(m_t \in X\} \) is a maximal-below-\(S'\) antichain. Therefore it suffices to show
that if $s \in S'$ is such that $\text{root}(S) \geq s$, then $s$ has infinitely many immediate successors in $S'$. First suppose that there is a $t \in I$ with $t \leq s$. Since members of $I$ are minimal, it follows that $(S')_s = S' \cap S_s$—hence $t \in S'$ and $m_t \in X$. Therefore $S'_t \subseteq S'$ and $s$ has infinitely many immediate successors in $S'$. If there is no such $t \in I$, then $S_s$ has no $<_0$-extension which decides a value of $A_n$ above $m$. Suppose now that $k \in \omega$ is such that $\max(|\{i | s^i \in S'_i\}) < k$. Let $S''$ be the $<_0$-extension of $S_s$ obtained by removing $\{t | (\exists i < k)s^i \leq t\}$. We claim that $\{m_t \in I \cap S''\}$ is infinite. Indeed, since $\{S_t \in I \cap S''\}$ is predense below $S''$, $S'' \models A_n \cap \{m_t \in I \cap S''\} \neq \emptyset$. If the set was finite then, by (*), $S''$ (hence $S_s$) would have $<_0$-extension picking one of the values. But now $X \cap \{m_t \in I \cap S''\}$ is nonempty, hence any $t \in I \cap S''$ with $m_t \in X$ is an extension of $s$ in $S'$—a contradiction to the choice of $k$.

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