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ON THE COHOMOLOGICAL DIMENSION OF THE LOCALIZATION FUNCTOR

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ABSTRACT. The left cohomological dimension of the localization functor is infinite for singular infinitesimal characters.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and X the flag variety of $\mathfrak{g}$, i.e. the variety of all Borel subalgebras in $\mathfrak{g}$. Let $\mathfrak{h}$ be the (abstract) Cartan algebra of $\mathfrak{g}$, Σ the root system in $\mathfrak{h}^*$ and Σ^+ the set of positive roots determined by the condition that the homogeneous line bundles $\mathscr{O}(-\mu)$ on X corresponding to dominant weights μ are positive. Denote by W the Weyl group of Σ. By a well-known result of Harish-Chandra the center $\mathscr{Z}(\mathfrak{g})$ of the enveloping algebra $\mathscr{U}(\mathfrak{g})$ is isomorphic to the Weyl group invariants $I(\mathfrak{h})$ in the symmetric algebra $S(\mathfrak{h})$. Therefore, the space of all maximal ideals in $\mathscr{Z}(\mathfrak{g})$ can be identified with the W-orbits in $\mathfrak{h}^*$. Let θ be such an orbit in $\mathfrak{h}^*$, and denote by J_θ the corresponding maximal ideal in $\mathscr{Z}(\mathfrak{g})$. Put $\mathscr{U}_\theta = \mathscr{U}(\mathfrak{g})/J_\theta\mathscr{U}(\mathfrak{g})$. Denote by $\mathscr{M}(\mathscr{U}_\theta)$ the category of $\mathscr{U}_\theta$-modules.

For any $\lambda \in \mathfrak{h}^*$, A. Beilinson and J. Bernstein defined a twisted sheaf of differential operators $\mathscr{D}_\lambda$ on X with the property that $\Gamma(X, \mathscr{D}_\lambda) = \mathscr{U}_\theta$ (compare [1], [6]). Denote by $\mathscr{M}_{\mathrm{qc}}(\mathscr{D}_\lambda)$ the category of quasicoherent $\mathscr{D}_\lambda$-modules on X. They also defined the *localization functor* Δ_λ from $\mathscr{M}(\mathscr{U}_\theta)$ into $\mathscr{M}_{\mathrm{qc}}(\mathscr{D}_\lambda)$ by the formula

$$\Delta_\lambda(V) = \mathscr{D}_\lambda \otimes_{\mathscr{U}_\theta} V$$

for a $\mathscr{U}_\theta$-module V.

Let $Q(\Sigma)$ be the root lattice in $\mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$, we denote by W_λ the subgroup of the Weyl group W given by $W_\lambda = \{w \in W | w\lambda - \lambda \in Q(\Sigma)\}$. Let $\Sigma^\vee$ be the root system in $\mathfrak{h}$ dual to Σ; and for any $\alpha \in \Sigma$, we denote by $\alpha^\vee \in \Sigma^\vee$ the dual root of α. Then, it is well-known that W_λ is the Weyl group of the root system $\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha^\vee(\lambda) \in \mathbf{Z}\}$. We define an order on Σ_λ by putting $\Sigma_\lambda^+ = \Sigma^+ \cap \Sigma_\lambda$. This defines a set of simple roots Π_λ of Σ_λ, and the corresponding set of simple reflections S_λ. Let l_λ be the length function on (W_λ, S_λ).

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We say that $\lambda \in \mathfrak{h}^*$ is *regular* if $\alpha^\vee(\lambda)$ is different from zero for any $\alpha \in \Sigma$ and that λ is *antidominant* if $\alpha^\vee(\lambda)$ is not a strictly positive integer for any $\alpha \in \Sigma^+$. We put $n(\lambda) = \min\{l_\lambda(w) | w \in W_\lambda, w\lambda \text{ is antidominant}\}$. Beilinson and Bernstein proved that, for *regular* λ, the left cohomological dimension of the localization functor is $\leq n(\lambda)$ ([2], [8]). In this note we prove the following complementary result.

Theorem. *Let $\lambda \in \mathfrak{h}^*$ be singular. Then the left cohomological dimension of the localization functor Δ_λ is infinite.*

Using the fact that the localization functor is an equivalence of the category $\mathcal{M}(\mathcal{U}_\theta)$ with the category $\mathcal{M}_{\mathrm{qc}}(\mathcal{D}_\lambda)$ for regular antidominant λ, Beilinson and Bernstein also proved that the homological dimension of $\mathcal{U}_\theta$ is $\leq \frac{1}{2}(\mathrm{Card}(\Sigma) + \mathrm{Card}(\Sigma_\lambda))$ if $\theta = W \cdot \lambda$ consists of regular elements (unpublished, compare [8]). On the contrary, our result immediately implies the following consequence.

Corollary. *If θ consists of singular elements in $\mathfrak{h}^*$, the homological dimension of $\mathcal{U}_\theta$ is infinite.*

This fact was observed earlier by A. Joseph and J. T. Stafford ([7, 4.20]). Our argument shows that this is a simple consequence of the analogous behavior of homological dimension for local rings.

Proof of the theorem. Let x be a point in X and denote by $\mathfrak{b}_x$ the corresponding Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ be its nilpotent radical. Let $\mathfrak{h}_x = \mathfrak{b}_x/\mathfrak{n}_x$. Then $\mathfrak{h}_x$ is canonically isomorphic to $\mathfrak{h}$ [6]. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_x$. Then the composition of the projection $\mathfrak{c} \to \mathfrak{h}_x$ with this map gives an isomorphism of $\mathfrak{c}$ onto $\mathfrak{h}$. The inverse map is called a *specialization* at x. For a $\mathcal{U}(\mathfrak{g})$-module V, we put

$$V_{\mathfrak{n}_x} = V/\mathfrak{n}_x V = \mathbf{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} V,$$

where we view $\mathbf{C}$ as a module with the trivial action of $\mathfrak{b}_x$. It has a natural structure of an $\mathfrak{h}_x$-module. Therefore, we can view it as an $\mathfrak{h}$-module. It follows that $V \to V_{\mathfrak{n}_x}$ is a right exact covariant functor from the category of $\mathcal{U}(\mathfrak{g})$-modules into the category of $\mathcal{U}(\mathfrak{h})$-modules. If we compose it with the forgetful functor into the category of vector spaces, we get the functor $H_0(\mathfrak{n}_x, -)$ of zeroth $\mathfrak{n}_x$-homology. By the Poincaré-Birkhoff-Witt theorem, free $\mathcal{U}(\mathfrak{g})$-modules are also $\mathcal{U}(\mathfrak{n}_x)$-free, what implies the equality for the left derived functors. Therefore, with some abuse of language, we view the $(-p)^{\mathrm{th}}$ left derived functor of $V \to V_{\mathfrak{n}_x}$ as the p^{th} $\mathfrak{n}_x$-homology functor $H_p(\mathfrak{n}_x, -) = \mathrm{Tor}_p^{\mathcal{U}(\mathfrak{n}_x)}(\mathbf{C}, -)$.

We need a technical result, which must be well known, but we were unable to find a reference.

1. **Lemma.** *$\mathcal{U}_\theta$ is free as $\mathcal{U}(\mathfrak{n}_x)$-module.*

Proof. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_x$. This determines a specialization of $\mathfrak{h}$ to $\mathfrak{c}$ and a nilpotent subalgebra $\bar{\mathfrak{n}}$ opposite to $\mathfrak{n}_x$. Then we have $\mathfrak{g} = \mathfrak{n}_x \oplus \mathfrak{c} \oplus \bar{\mathfrak{n}}$ and $\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}})$ as a left $\mathscr{U}(\mathfrak{n}_x)$-module for left multiplication. Let $\mathrm{F}_p\mathscr{U}(\mathfrak{c})$, $p \in \mathbf{Z}_+$, be the degree filtration of $\mathscr{U}(\mathfrak{c})$. Then we define a filtration $\mathrm{F}_p\mathscr{U}(\mathfrak{g})$, $p \in \mathbf{Z}_+$, of $\mathscr{U}(\mathfrak{g})$ via

$$\mathrm{F}_p\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} \mathrm{F}_p\mathscr{U}(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}}).$$

This is clearly a $\mathscr{U}(\mathfrak{n}_x)$-module filtration. The corresponding graded module is

$$\operatorname{Gr}\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} S(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}}).$$

This filtration induces a filtration on the submodule $J_\theta\mathscr{U}(\mathfrak{g})$ and the quotient module $\mathscr{U}_\theta$. The Harish-Chandra homomorphism $\gamma\colon \mathscr{Z}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{h})$ is compatible with the degree filtrations and the homomorphism $\operatorname{Gr}\gamma$ is an isomorphism of $\operatorname{Gr}\mathscr{Z}(\mathfrak{g})$ onto the subalgebra $I(\mathfrak{h})$ of all W-invariants in $S(\mathfrak{h})$ ([4, Ch. VIII, §8, no. 5]). Denote by $I_+(\mathfrak{h})$ the homogeneous ideal spanned by the elements of strictly positive degree in $I(\mathfrak{h})$. Then

$$\operatorname{Gr} J_\theta\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} I_+(\mathfrak{c})S(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}}).$$

It follows that

$$\begin{aligned}\operatorname{Gr}\mathscr{U}_\theta &= (\operatorname{Gr}\mathscr{U}(\mathfrak{g}))/(\operatorname{Gr} J_\theta\mathscr{U}(\mathfrak{g}))\\ &= (\mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} S(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}}))/(\mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} I_+(\mathfrak{c})S(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}}))\\ &= \mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} (S(\mathfrak{c})/(I_+(\mathfrak{c})S(\mathfrak{c}))) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}}),\end{aligned}$$

i.e. it is a free $\mathscr{U}(\mathfrak{n}_x)$-module. Moreover, by ([4, Ch. V, §5, no. 2, Th. 1]) we know that the dimension of the complex vector space $S(\mathfrak{h})/(I_+(\mathfrak{h})S(\mathfrak{h}))$ is $\operatorname{Card} W$. It follows that $\mathscr{U}_\theta$ has a finite filtration by $\mathscr{U}(\mathfrak{n}_x)$-submodules such that $\operatorname{Gr}\mathscr{U}_\theta$ is a free $\mathscr{U}(\mathfrak{n}_x)$-module. By induction in length, this implies that $\mathscr{U}_\theta$ is a free $\mathscr{U}(\mathfrak{n}_x)$-module. □

Let ρ be the half-sum of roots in Σ^+. Denote by $\varphi\colon \mathscr{U}(\mathfrak{h}) \to \mathscr{U}(\mathfrak{h})$ the automorphism given by $\varphi(\xi) = \xi + \rho(\xi)$ for $\xi \in \mathfrak{h}$. Then, $\varphi(\gamma(\mathscr{Z}(\mathfrak{g})))$ is the algebra of W-invariants in $\mathscr{U}(\mathfrak{h})$. In addition, as we remarked in the preceding proof, the dimension of the vector space $\mathscr{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\mathscr{U}(\mathfrak{h})$ is equal to $\operatorname{Card} W$. This implies that $V_\theta = \mathscr{U}(\mathfrak{h})/\gamma(J_\theta)\mathscr{U}(\mathfrak{h})$ is an $\mathscr{U}(\mathfrak{h})$-module of dimension $\dim_{\mathbf{C}} V_\theta = \operatorname{Card} W$.

For $\mu \in \mathfrak{h}^*$, we denote by I_μ the corresponding maximal ideal in $\mathscr{U}(\mathfrak{h})$.

2. **Lemma.** *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W\cdot\lambda$. Then:*

(i) *V_θ is a $\mathscr{U}(\mathfrak{h})$-module of dimension* $\dim_{\mathbf{C}} V_\theta = \operatorname{Card} W$;

(ii) *the characteristic polynomial of the action of $\xi \in \mathfrak{h}$ on V_θ is*

$$P(\xi) = \prod_{w\in W} (\xi - (w\lambda + \rho)(\xi));$$

(iii) *$H_0(\mathfrak{n}_x, \mathscr{U}_\theta)$ is a direct sum of countably many copies of V_θ.*

Proof. We already established (i). Clearly, $I_\mu \supset \varphi(\gamma(J_\theta))\mathscr{U}(\mathfrak{h})$ is equivalent to $\mu = w\lambda$ for some $w \in W$. Hence the linear transformation of $\mathscr{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\mathscr{U}(\mathfrak{h})$ induced by multiplication by ξ has eigenvalues $(w\lambda)(\xi)$, $w \in W$, and by symmetry they all have the same multiplicity. This in turn implies that

$$\varphi(P(\xi)) = \prod_{w\in W} \varphi(\xi - (w\lambda + \rho)(\xi)) = \prod_{w\in W} (\xi - (w\lambda)(\xi))$$

is the characteristic polynomial for the action of ξ on $\mathscr{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\mathscr{U}(\mathfrak{h})$. This proves (ii).

(iii) As in the proof of 1, we fix a specialization $\mathfrak{c}$ of $\mathfrak{h}$ and choose a nilpotent subalgebra $\bar{\mathfrak{n}}$ opposite to $\mathfrak{n}_x$. By the Poincaré-Birkhoff-Witt theorem, it follows that as a vector space $\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}})$. Moreover,

$$H_0(\mathfrak{n}_x, \mathscr{U}_\theta) = \mathscr{U}(\mathfrak{g})/(J_\theta \mathscr{U}(\mathfrak{g}) + \mathfrak{n}_x \mathscr{U}(\mathfrak{g})).$$

Denote by $\gamma_x : \mathscr{Z}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{c})$ the composition of the specialization map with the Harish-Chandra homomorphism γ. Then

$$J_\theta\mathscr{U}(\mathfrak{g}) + \mathfrak{n}_x\mathscr{U}(\mathfrak{g}) = J_\theta\mathscr{U}(\mathfrak{c})\mathscr{U}(\bar{\mathfrak{n}}) + \mathfrak{n}_x\mathscr{U}(\mathfrak{g}) = \gamma_x(J_\theta)\mathscr{U}(\mathfrak{c})\mathscr{U}(\bar{\mathfrak{n}}) + \mathfrak{n}_x\mathscr{U}(\mathfrak{g}),$$

which implies that under the above isomorphism

$$J_\theta\mathscr{U}(\mathfrak{g}) + \mathfrak{n}_x\mathscr{U}(\mathfrak{g}) = (\mathbf{C} \otimes_{\mathbf{C}} \gamma_x(J_\theta)\mathscr{U}(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}})) \oplus (\mathfrak{n}_x\mathscr{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{c}) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}})).$$

This yields

$$H_0(\mathfrak{n}_x, \mathscr{U}_\theta) = \mathscr{U}(\mathfrak{c})/(\gamma_x(J_\theta)\mathscr{U}(\mathfrak{c})) \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}}) = V_\theta \otimes_{\mathbf{C}} \mathscr{U}(\bar{\mathfrak{n}})$$

and the action of $\mathfrak{h}$ is given by multiplication in the first factor. □

Therefore, maximal ideals in the ring V_θ are the images of the maximal ideals $I_{w\lambda+\rho}$, $w \in W$, in $\mathscr{U}(\mathfrak{h})$, under the quotient map $\mathscr{U}(\mathfrak{h}) \to V_\theta$.

Let $W(\lambda)$ be the stabilizer of λ in W. Denote by $R_{w\lambda}$ the localization of V_θ at $I_{w\lambda+\rho}$. Then, by ([3, Ch. IV, §2, no. 5, Cor. 1 of Prop. 9]), we have

$$V_\theta = \prod_{w\in W/W(\lambda)} R_{w\lambda}.$$

Since the local rings $R_{w\lambda}$ are finite-dimensional, they are regular if and only if $\dim_{\mathbf{C}} R_{w\lambda} = 1$. Because of the Weyl group symmetry these rings are isomorphic, hence $\dim_{\mathbf{C}} R_{w\lambda} = \operatorname{Card} W(\lambda)$ for any $w \in W$. This finally leads to the following critical observation.

3. **Corollary.** *The following conditions are equivalent*:

(i) λ *is regular*;

(ii) *the rings* $R_{w\lambda}$, $w \in W$, *are regular local rings.*

By 1, we can calculate $\mathfrak{n}_x$-homology of V using a left resolution of V by free $\mathscr{U}_\theta$-modules. Therefore, we can view $H_p(\mathfrak{n}_x, V)$ as V_θ-modules. Also, for any $\lambda \in \theta, \mathbf{C}_{\lambda+\rho} = \mathscr{U}(\mathfrak{h})/I_{\lambda+\rho}$ is a V_θ-module.

For any $\mathscr{O}_X$-module $\mathscr{F}$ on X we denote by $T_x(\mathscr{F})$ its geometric fibre at x, i.e. $T_x(\mathscr{F}) = \mathbf{C} \otimes_{\mathscr{O}_x} \mathscr{F}_x$, where $\mathscr{O}_x$ is the local ring of X at x. Since X is a smooth algebraic variety, $\mathscr{O}_x$ is a regular local ring. Hence, the left cohomological dimension of the right exact functor T_x is $\leq \dim X$.

For any abelian category $\mathscr{A}$, denote by $D^-(\mathscr{A})$ the derived category of $\mathscr{A}$-complexes bounded from above, and by D the natural imbedding of $\mathscr{A}$ into $D^-(\mathscr{A})$ which maps an object V of $\mathscr{A}$ into the complex $D(V)$ such that $D(V)^p = 0$ for $p \neq 0$ and $D(V)^0 = V$.

Since the localization functor Δ_λ is right exact, it defines the left derived functor $L\Delta_\lambda$ from $D^-(\mathscr{U}_\theta)$ into $D^-(\mathscr{D}_\lambda)$. Analogously, T_x defines the left derived functor LT_x from $D^-(\mathscr{D}_\lambda)$ into the derived category $D^-(\mathbf{C})$ of complexes of complex vector spaces bounded from above.

4. **Proposition.** *Let $\lambda \in \mathfrak{h}^*$, $\theta = W\cdot\lambda$ and $x \in X$. Then the functors $LT_x \circ L\Delta_\lambda$ and $D(\mathbf{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbf{C}) \overset{L}{\otimes}_{\mathscr{U}(\mathfrak{n}_x)} -)$ from $D^-(\mathscr{U}_\theta)$ into $D^-(\mathbf{C})$ are isomorphic.*

Proof. By 1, we know that $\mathscr{U}_\theta$ is acyclic for the functor $H_0(\mathfrak{n}_x, -) = \mathbf{C}\otimes_{\mathscr{U}(\mathfrak{n}_x)} -$. By 2, we also know that $\mathbf{C} \otimes_{\mathscr{U}(\mathfrak{n}_x)} \mathscr{U}_\theta$ is acyclic for the functor $\mathbf{C}_{\lambda+\rho} \otimes_{V_\theta} -$. Let $F^\cdot$ be a complex isomorphic to $V^\cdot$ consisting of free $\mathscr{U}_\theta$-modules. Then, since the functors commute with infinite direct sums, we get

$$D(\mathbf{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbf{C}) \overset{L}{\otimes}_{\mathscr{U}(\mathfrak{n}_x)} V^\cdot) = \mathbf{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathbf{C} \otimes_{\mathscr{U}(\mathfrak{n}_x)} F^\cdot).$$

On the other hand, the localization $\Delta_\lambda(\mathscr{U}_\theta) = \mathscr{D}_\lambda$ is a locally free $\mathscr{O}_X$-module, and therefore acyclic for T_x. This implies that

$$LT_x(L\Delta_\lambda(V^\cdot)) = T_x(\Delta_\lambda(F^\cdot)).$$

Hence, to complete the proof it is enough to establish the following identity

$$T_x(\Delta_\lambda(\mathscr{U}_\theta)) = \mathbf{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathbf{C} \otimes_{\mathscr{U}(\mathfrak{n}_x)} \mathscr{U}_\theta).$$

First, we have $T_x(\Delta_\lambda(\mathscr{U}_\theta)) = T_x(\mathscr{D}_\lambda)$. Moreover, from the construction of $\mathscr{D}_\lambda$ ([1], [6]) and the properties of the Harish-Chandra homomorphism, it follows that

$$\begin{aligned}
T_x(\mathscr{D}_\lambda) &= (\mathscr{U}(\mathfrak{g})/\mathfrak{n}_x\mathscr{U}(\mathfrak{g}))/(I_{\lambda+\rho}(\mathscr{U}(\mathfrak{g})/\mathfrak{n}_x\mathscr{U}(\mathfrak{g}))) \\
&= \mathbf{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathscr{U}(\mathfrak{g})/\mathfrak{n}_x\mathscr{U}(\mathfrak{g}))/(\gamma(J_\theta)(\mathscr{U}(\mathfrak{g})/\mathfrak{n}_x\mathscr{U}(\mathfrak{g}))) \\
&= \mathbf{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathscr{U}(\mathfrak{g})/(J_\theta\mathscr{U}(\mathfrak{g}) + \mathfrak{n}_x\mathscr{U}(\mathfrak{g}))) = \mathbf{C}_{\lambda+\rho} \otimes_{V_\theta} H_0(\mathfrak{n}_x, \mathscr{U}_\theta). \quad \square
\end{aligned}$$

5. *Remark.* Using spectral sequences instead of derived categories, 4 implies the following statement: The Grothendieck spectral sequences for composition of derived functors with E_2-terms $E_2^{p,q} = L^pT_x(L^q\Delta_\lambda(V))$ and $E_2'^{p,q} = \mathrm{Tor}_{-p}^{V_\theta}(\mathbf{C}_{\lambda+\rho}, H_{-q}(\mathfrak{n}_x, V))$ converge to the same limit.

To prove the theorem it is enough to establish the following fact.

6. **Lemma.** *Let* $\lambda \in \mathfrak{h}^*$ *be singular. Then there exists* $V \in \mathcal{M}(\mathcal{U}_\theta)$ *such that* $L\Delta_\lambda(D(V))$ *is not a cohomologically bounded complex.*

Proof. Since the functor T_x has finite left cohomological dimension, it is enough to find a $\mathcal{U}_\theta$-module V such that $LT_x(L\Delta_\lambda(V))$ is not a bounded complex for some $x \in X$. By 4, this is equivalent to the fact that $D(\mathbf{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbf{C}) \overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)} D(V))$ is not a cohomologically bounded complex.

Let w_0 be the longest element in W. Fix a Borel subalgebra $\mathfrak{b}_0$, and consider the Verma module $M(w_0\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbf{C}_{w_0\lambda-\rho}$. Pick x so that $\mathfrak{b}_x$ is opposite to $\mathfrak{b}_0$. Then, by the Poincaré-Birkhoff-Witt theorem, $M(w_0\lambda)$ is isomorphic to $\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbf{C}} \mathbf{C}_{w_0\lambda-\rho}$ as $\mathcal{U}(\mathfrak{n}_x)$-module. This implies, since $\mathfrak{b}_x$ is opposite to $\mathfrak{b}_0$ and corresponding specializations differ by w_0, that

$$H_0(\mathfrak{n}_x, M(w_0\lambda)) = \mathbf{C}_{\lambda+\rho},$$

and $H_p(\mathfrak{n}_x, M(w_0\lambda)) = 0$ for $p \in \mathbf{N}$. Therefore,

$$D(\mathbf{C}) \overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)} D(M(w_0\lambda)) = D(\mathbf{C}_{\lambda+\rho}),$$

and

$$D(\mathbf{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbf{C}) \overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)} D(M(w_0\lambda))) = D(\mathbf{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} D(\mathbf{C}_{\lambda+\rho}).$$

Clearly, we have

$$H^{-p}(D(\mathbf{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} D(\mathbf{C}_{\lambda+\rho})) = \operatorname{Tor}_p^{V_\theta}(\mathbf{C}_{\lambda+\rho}, \mathbf{C}_{\lambda+\rho}), \qquad p \in \mathbf{Z}_+.$$

On the other hand, we have

$$\operatorname{Tor}_p^{V_\theta}(\mathbf{C}_{\lambda+\rho}, \mathbf{C}_{\lambda+\rho}) = \operatorname{Tor}_p^{R_\lambda}(\mathbf{C}, \mathbf{C}), \qquad p \in \mathbf{Z}_+.$$

Since R_λ is not a regular local ring by 3, its homological dimension is infinite ([5, 17.3.1]) and $\operatorname{Tor}_p^{R_\lambda}(\mathbf{C}, \mathbf{C}) \neq 0$ for $p \in \mathbf{Z}_+$ ([5, 17.2.11]). □

This completes the proof of the theorem.

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