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RELATIVELY OPEN MAPPINGS

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ABSTRACT. A bounded linear operator on a Banach space which is one-one, dense and 'relatively almost open' must be invertible.

INTRODUCTION

In Berberian's book [1] it is shown that, on a reflexive Banach space X, a non-invertible operator $T \in L(X)$ which is neither a left nor a right zero-divisor must be both a topological left and a topological right zero-divisor in the Banach algebra $L(X)$ ([1, Corollary 57.11]). In this note we show that the space X need not be reflexive, and further strengthen the statement.

If $T: X \to Y$ is a bounded linear operator between normed spaces write $T^\frown: X \to T(X)$ for its 'truncation': thus $T^\frown$ is automatically onto, and

$$(0.1) \qquad T \text{ one-one} \iff T^\frown \text{ one-one}$$

and

$$(0.2) \qquad T \text{ bounded below} \iff T^\frown \text{ bounded below.}$$

We may refer the reader to [1], [2] or [3] for the concepts of "bounded below", "open" and "almost open": we shall call $T \in L(X, Y)$ *relatively open* if $T^\frown$ is open and call T *relatively almost open* if $T^\frown$ is almost open. The second of these concepts can be expressed in terms of the first, via duality; writing $T^*: Y^* \to X^*$ for the dual or *adjoint* of $T: X \to Y$, we have

Theorem 1. *If $T \in L(X, Y)$ is a bounded linear operator between normed spaces then*

$$(1.1) \qquad T \textit{ relatively almost open} \iff T^* \textit{ relatively open.}$$

Proof. We have, by the definition of "relatively almost open" and [2,(2.3.4)], T relatively almost open $\iff T^\frown$ almost open $\iff (T^\frown)^*$ bounded below and, by the definition of "relatively open",

$$T^* \text{ relatively open} \iff (T^*)^\frown \text{ open};$$

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if $(T^\frown)^*$ is bounded below then $\|g^\circ\|_{T(X)^*} \le k\|g^\circ T\|_{X^*}$ for all $g^\circ \in T(X)^*$; now for arbitrary $f = gT \in T^*(Y^*)$ use the Hahn-Banach Theorem to extend the restriction g° of g to $T(X)$ to $h \in Y^*$ with $\|h\|_{Y^*} = \|g^\circ\|_{T(X)^*}$ to obtain $f = gT \in \{hT : \|h\|_{Y^*} \le k\|f\|_{X^*}\}$, which says that $(T^*)^\frown$ is open. Conversely if this is assumed then $\|g\|_{T(X)^*} = \|h\|_{T(X)^*} \le \|h\|_{Y^*} \le k\|gT\|_{X^*}$.

Our main result is

Theorem 2. *If $T \in L(X)$ for a Banach space X then*

(2.1) $$T \text{ one-one, dense and relatively almost open} \implies T \text{ invertible.}$$

Proof. Since

$$T \text{ dense} \iff T^* \text{ one-one}$$

we have, by (1.1),

$$T \text{ dense and relatively almost open} \implies T^* \text{ one-one and relatively open},$$

which means T^* is bounded below; thus T is almost open (cf. [2, (2.3.4)]); quoting half the open mapping theorem ([4, Lemma 5.2.2]) we can conclude that T is open.

To see how this relates to Berberian's result, recall that the *divisors of zero* in the Banach algebra $L(X)$ are the operators which are not one-one, or not dense, while the *topological zero-divisors* are those which are not bounded below, or not almost open:

Theorem 3. *If $T \in L(X)$ is not relatively almost open then it is both a topological left and a topological right zero-divisor.*

Proof. If T is not relatively almost open then it is not almost open, and it is not relatively open, therefore not bounded below.

The converse of Theorem 3 fails (consider the operator 0).

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