RELATIVELY OPEN MAPPINGS

WOO YOUNG LEE

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Abstract. A bounded linear operator on a Banach space which is one-one,
dense and 'relatively almost open' must be invertible.

Introduction

In Berberian's book [1] it is shown that, on a reflexive Banach space $X$, a
non-invertible operator $T \in L(X)$ which is neither a left nor a right zero-
divisor must be both a topological left and a topological right zero-divisor in
the Banach algebra $L(X)$ ([1, Corollary 57.11]). In this note we show that the
space $X$ need not be reflexive, and further strengthen the statement.

If $T: X \to Y$ is a bounded linear operator between normed spaces write
$T^\sim: X \to T(X)$ for its 'truncation': thus $T^\sim$ is automatically onto, and

\begin{equation}
T \text{ one-one} \iff T^\sim \text{ one-one}
\end{equation}

and

\begin{equation}
T \text{ bounded below} \iff T^\sim \text{ bounded below}.
\end{equation}

We may refer the reader to [1], [2] or [3] for the concepts of “bounded below”,
“open” and “almost open”: we shall call $T \in L(X, Y)$ relatively open if $T^\sim$
is open and call $T$ relatively almost open if $T^\sim$ is almost open. The second
of these concepts can be expressed in terms of the first, via duality; writing
$T^*: Y^* \to X^*$ for the dual or adjoint of $T: X \to Y$, we have

Theorem 1. If $T \in L(X, Y)$ is a bounded linear operator between normed spaces
then

\begin{equation}
T \text{ relatively almost open} \iff T^* \text{ relatively open}.
\end{equation}

Proof. We have, by the definition of “relatively almost open” and [2,(2.3.4)],
$T$ relatively almost open $\iff T^\sim$ almost open $\iff (T^\sim)^*$ bounded below
and, by the definition of “relatively open”,

\begin{equation}
T^* \text{ relatively open} \iff (T^*)^\sim \text{ open}.
\end{equation}
if \((T^\sim)^*\) is bounded below then \(\|g^\circ\|_{T(X)} \leq k\|g^\circ T\|_X\) for all \(g^\circ \in T(X)^*\); now for arbitrary \(f = gT \in T^*(Y^*)\) use the Hahn-Banach Theorem to extend the restriction \(g^\circ\) of \(g\) to \(T(X)\) to \(h \in Y^*\) with \(\|h\|_Y = \|g^\circ\|_{T(X)}\) to obtain \(f = gT \in \{hT : \|h\|_Y \leq k\|f\|_X\}\), which says that \((T^*)^\sim\) is open. Conversely if this is assumed then \(\|g\|_{T(X)} = \|h\|_{T(X)} \leq \|h\|_Y \leq k\|gT\|_X\).

Our main result is

**Theorem 2.** If \(T \in L(X)\) for a Banach space \(X\) then

\[(2.1) \quad \text{T one-one, dense and relatively almost open} \implies \text{T invertible.}\]

**Proof.** Since

\[T \text{ dense } \iff T^* \text{ one-one}\]

we have, by (1.1),

\(T\) dense and relatively almost open \(\implies T^*\) one-one and relatively open,

which means \(T^*\) is bounded below; thus \(T\) is almost open (cf. [2, (2.3.4)]); quoting half the open mapping theorem ([4, Lemma 5.2.2]) we can conclude that \(T\) is open.

To see how this relates to Berberian's result, recall that the **divisors of zero** in the Banach algebra \(L(X)\) are the operators which are not one-one, or not dense, while the **topological zero-divisors** are those which are not bounded below, or not almost open:

**Theorem 3.** If \(T \in L(X)\) is not relatively almost open then it is both a topological left and a topological right zero-divisor.

**Proof.** If \(T\) is not relatively almost open then it is not almost open, and it is not relatively open, therefore not bounded below.

The converse of Theorem 3 fails (consider the operator 0).

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**References**


**Department of Mathematics, Sung Kyun Kwan University, Seoul 110-745, Korea**