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GAUSSIAN CURVATURES OF LORENTZIAN METRICS ON THE PLANE AND PUNCTURED PLANES

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Dedicated to Professors Gu Chaohao and Hu Hesheng

ABSTRACT. We prove that every $f \in C^k(R^2)$ is the Gaussian curvature of some C^{k+1}-Lorentzian metric $(0 \le k \le \infty)$. Let M denote the cylinder. We prove that every continuous function on M is the Gaussian curvature of some C^1-Lorentzian metric. If $f \in C^k(M)$ satisfies the condition (H) in the Lemma 2 below, then it is the curvature function of some C^{k+1}-Lorentzian metric. If $f \in C^k(R^2)$ $(1 \le k \le \infty)$ has compact support, then the Lorentzian metric can be made complete.

1. INTRODUCTION

Given a function on the surface, does there exist a metric whose Gaussian curvature is the given function? This is to solve the second order nonlinear partial differential equation:

$$K(l) = f, \tag{1}$$

where f is the given function, $K(l)$ is the Gaussian curvature of the definite or indefinite metric l. Many mathematicians have studied the case when l is a Riemannian metric. For the Lorentzian metric l, Burns [1] got some results in 1977. In this note we are going to solve (1) on the 2-dimensional plane and to make a try on punctured planes (skirts, shirts, T-shirts, etc.).

2. CURVATURE FUNCTIONS ON THE PLANE

Lemma 1. *Suppose $h \in L^1(R^2)$, $\|h\|_{L^1(\mathbf{R}^2)} < 1/\pi$; then*

$$w(x,y) = \int_0^x \int_0^y h(s,t)e^{w(s,t)}ds\,dt \qquad on\ R^2 \tag{2}$$

admits a unique solution $w \in C(R^2)$.

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Proof. Define a sequence by

$$u_0 = 0,$$
$$u_{n+1}(x,y) = \int_0^x \int_0^y h(s,t)e^{u_n(s,t)}ds\,dt, \qquad n = 0, 1, 2, \dots;$$

then $\|u_n\|_{C(R^2)} < 1$, $\|u_{n+1} - u_n\|_{C(R^2)} < (e/\pi)\|u_n - u_{n-1}\|_{C(R^2)}$. Hence $\{u_n\}$ converges uniformly on R^2 and the limit function is a solution of the equation (2). The uniqueness is obvious. Q.E.D.

Theorem 1. *Let k be a nonnegative integer or $k = \infty$. Suppose $f \in C^k(R^2)$. Then there exists a C^{k+1}-Lorentzian metric on R^2 which is pointwise conformal to the standard flat Lorentzian metric $dx\,dy$ such that its Gaussian curvature equals f.*

Proof. Let $G \in C^\infty(R)$ such that, for all $t \in R$

(3) $$G(t) > \max\{\pi(|f(x,y)| + 1) \mid |x| \le |y| = |t| \text{ or } |y| \le |x| = |t|\}.$$

Let

$$h(x,y) = -\frac{1}{2}\frac{f(x,y)}{G(x)G(y)}e^{-x^2-y^2};$$

then

$$\|h\|_{L^1(R^2)} < \frac{1}{\pi}.$$

By Lemma 1, there exists $w \in C(R^2)$ satisfying the equation (2). Then

$$u(x,y) = w(x,y) - x^2 - y^2 - \ln(G(x)G(y))$$

satisfies

$$u_{xy} = -\frac{1}{2}f(x,y)e^u \qquad \text{on } R^2,$$

which means the metric $e^u dx\,dy$ has the Gaussian curvature f. Q.E.D.

As a consequence, given a C^k-function f on any 2-manifold M, for each point $p \in M$, locally there always exists a C^{k+1}-Lorentzian metric whose Gaussian curvature equals f in a neighborhood of p.

3. Curvature functions on a punctured plane

Lemma 2. *Let k be a nonnegative integer or $k = \infty$, $p = (x_0, y_0) \in R^2$. Suppose $f \in C^k(R^2 \backslash \{p\})$ satisfies the following condition:*

(H) *For each n $(0 \le n < k+1)$, there exists a function $F \in L^1[-1,1]$ (set $F(0) = +\infty$), such that, for $(x,y) \in [-1,1]^2 \backslash \{(0,0)\}$,*

$$\left|\frac{\partial}{\partial x^n}f(x+x_0, y+y_0)\right| + \left|\frac{\partial}{\partial y^n}f(x+x_0, y+y_0)\right| < \min\{F(x), F(y)\}.$$

Then any solution of the equation (2) (*replacing* h *by* f) is in fact in $C^{k+1}(R^2\backslash\{p\})$.

Proof. By Lebesgue's dominated convergence theorem. Q.E.D.

Theorem 2. *Let* k *be a nonnegative integer or* $k = \infty$. $p = (0,0) \in R^2$. *Suppose* $f \in C^k(R^2\backslash\{p\})$ *satisfies the condition* (*H*) *of Lemma* 2; *then there exists a* C^{k+1}*-Lorentzian metric on* $R^2\backslash\{p\}$ *such that its Gaussian curvature equals* f.

Proof. There is $G \in C^\infty(R)$ satisfying (3) for $|t| > 1$. Then

$$g(x,y) = -\frac{f(x,y)}{2G(x)G(y)}e^{-x^2-y^2}$$

is integrable on R^2. Pick $\varepsilon > 0$ such that

$$h = \varepsilon g$$

satisfies $\|h\|_{L^1(R^2)} < 1/\pi$. Clearly h also satisfies the condition (H) of Lemma 2. Hence the solution w to the equation (2) (in Lemma 1) is in $C^{k+1}(R^2\backslash\{p\})$ by Lemma 2, and so is the function u which is defined by

$$u(x,y) = w(x,y) - x^2 - y^2 - \ln(G(x)G(y)) + \ln\varepsilon .$$

Then the Lorentzian metric $e^u dx\,dy$, which has the Gaussian curvature f, is of class C^{k+1}. Q.E.D.

Remark 1. Let $M = R^2\backslash\{\text{finite points}\}$, $f \in C^k(M)$. If for each $(x_0, y_0) \in R^2\backslash M$, f satisfies the condition (H) of Lemma 2, then the same conclusion holds.

Theorem 3. *On a cylinder, every continuous function is the Gaussian curvature for some* C^1*-Lorentzian metric.*

Proof. Look at a cylinder as $R^2\backslash\{p\}$, say, $p = (0,0)$. Suppose $f \in C(R^2\backslash\{p\})$. We first shrink f into control. Pick $g \in C(R^2\backslash\{(0,0)\})$, such that

$$g(x,y) = h(r) \text{ depends only on } r = \sqrt{x^2+y^2},$$
$$g > f \text{ on } R^2\backslash\{(0,0)\},$$
$$h(r) \text{ is decreasing for } r \in (0,2].$$

Let φ be a C^∞-diffeomorphism of $(0, +\infty)$ such that $\varphi|_{(1,+\infty)} = \text{id}$ and $h(\varphi) \in L^1(0,1]$. Define

$$\Phi\colon R^2\backslash\{(0,0)\} \to R^2\backslash\{(0,0)\},$$
$$\Phi(x,y) = \left(\frac{\varphi(r)}{r}x, \frac{\varphi(r)}{r}y\right),$$

where $r = \sqrt{x^2+y^2}$. Define

$$F(t) = h(\varphi(|t|)) \qquad \text{for } t \neq 0, \quad \text{and } F(0) = +\infty.$$

Then $F \in L^1[-1,1]$ and for $(x,y) \in [-1,1]^2 \backslash \{(0,0)\}$,

$$f(\Phi(x,y)) = f\left(\frac{\varphi(r)}{r}x, \frac{\varphi(r)}{r}y\right) < g\left(\frac{\varphi(r)}{r}x, \frac{\varphi(r)}{r}y\right)$$
$$= h(\varphi(r)) = F(r) \leq \min(F(x), F(y)).$$

Applying Theorem 2 we know that $f(\Phi)$ is the curvature function of some C^1-Lorentzian metric l. Hence the pull-back metric $(\Phi^{-1})^* l$ has the curvature f. Q.E.D.

Remark 2. The same conclusion holds for the manifold $R^2 \backslash A$ where A is a discrete subset of R^2.

4. Curvature functions for complete Lorentzian metrics

Throughout this section, we use the notation $B_r = \{x^2 + y^2 \leq r^2\}$. For the Lorentzian metric $e^u dx\,dy$ on R^2, the equations of the geodesic are

$$\ddot{x} + u_x \dot{x}^2 = 0,$$
$$\ddot{y} + u_y \dot{y}^2 = 0.$$

In particular, the characteristic lines (i.e. those lines parallel to the x-axis or y-axis) are geodesics. We say a geodesic $\gamma(t)$ is forward complete (complete, respectively) if it exists for all $t \geq 0$ (all t, resp.). Note the fact that for a positive function $f \in C(R)$, the solution to the problem

$$\dot{x} = f(x), \qquad x(0) = x_0$$

exists for all $t \geq 0$ if and only if

$$\int_{x_0}^{+\infty} \frac{dx}{f(x)} = +\infty.$$

This implies

Lemma 3. *Let $u \in C(R^2)$. The characteristic lines of the metric $e^u dx\,dy$ are complete if and only if for any x and y,*

$$(4) \quad \int_0^\infty e^{u(x,y)}\,dx = \int_0^\infty e^{u(x,y)}dy = \int_{-\infty}^0 e^{u(x,y)}dx = \int_{-\infty}^0 e^{u(x,y)}dy = +\infty.$$

Proof. Consider the characteristic line $\gamma(t) = (x(t), y_0)$. It is a geodesic if properly parametrized:

$$\ddot{x}(t) = -u_x(x(t), y_0)\dot{x}(t)^2, \qquad x(0) = x_0, \qquad \dot{x}(0) \neq 0.$$

Let

$$p(t) = \dot{x}(t);$$

then

$$\frac{dp}{dx} = -u_x(x, y_0)p.$$

Therefore

$$p = ce^{-u(x,y_0)} \qquad (c = \dot{x}(0)e^{u(x_0,y_0)} \neq 0).$$

That is

$$\frac{dx}{dt} = ce^{-u(x,y_0)}.$$

Hence $\gamma(t)$ is forward complete if and only if $x(t)$ is , if and only if

$$\int_{x_0}^{\infty} e^{u(x,y_0)}dx = +\infty \qquad (\text{if } c > 0)$$

or

$$\int_{-\infty}^{x_0} e^{u(x,y_0)}dx = +\infty \qquad (\text{if } c < 0).$$

All characteristic lines are complete if and only if every characteristic line is forward complete, if and only if (4) holds. Q.E.D.

Remark 3. By Lemma 3 one can see that the metric constructed in the proof of Theorem 1 is not complete.

Corollary 1. *If* $u \in C(R^2)$ *satisfies*

$$\liminf_{r\to\infty R\geq r} (\|u\|_{C(B_R)} - \ln R) < +\infty,$$

then every characteristic line of the Lorentzian metric $e^u dx\,dy$ *on* R^2 *is complete.*

Proof. It is equivalent to prove that the integrals in Lemma 3 are $+\infty$. From the condition, there exist a sequence $r_n \to +\infty$ and a constant $c > 0$ such that

$$\|u\|_{C(B_{r_n})} - \ln r_n \leq \ln c.$$

Then

$$(5) \qquad e^u \geq \frac{1}{cr_n} \qquad \text{on } B_{r_n}.$$

Fix y. Define

$$x_n = \sqrt{r_n^2 - y^2}.$$

By passing to a subsequence we may assume all $r_n > |y|$ and

$$(6) \qquad \frac{x_n - x_{n-1}}{r_n} > \frac{1}{2} \qquad \text{for all } n.$$

Then

$$\begin{aligned}\int_0^{\infty} e^{u(x,y)}dx &\geq \sum_n \int_{x_{n-1}}^{x_n} e^{u(x,y)}dx \\ &\geq \sum_n \frac{1}{c}\frac{x_n - x_{n-1}}{r_n} \qquad (\text{by } (5)) \\ &= +\infty \qquad (\text{by } (6)).\end{aligned}$$

Similarly the other three integrals also equal $+\infty$. Then Lemma 3 applies. Q.E.D.

Theorem 4. *If* $u \in C^{1,\alpha}(R^2)$ $(0 < \alpha < 1)$ *satisfies*

$$\liminf_{r\to\infty R\geq r}(\|u\|_{C^1(B_R)} - \frac{1}{2}\ln\ln R) = -\infty \tag{7}$$

then $e^u dx\,dy$ *is a complete Lorentzian metric on* R^2.

Proof. By Corollary 1, every characteristic line is complete. Now consider an arbitrary noncharacteristic geodesic $\gamma(t) = (x(t), y(t))$ $(\dot{x}(t)\dot{y}(t) \neq 0)$. It suffices to prove that γ is forward complete. Define the notations

$$p(t) = \dot{x}(t), q(t) = \dot{y}(t),$$
$$\alpha(t) = \frac{1}{p(t)}, \beta(t) = \frac{1}{q(t)},$$
$$X(t) = (x(t), y(t), \alpha(t), \beta(t)),$$
$$\|(x, y, z, t)\| = \sqrt{x^2 + y^2 + z^2 + t^2},$$
$$\|X\|_{C[0,T]} = \sup_{t\in[0,T]} \|X(t)\|.$$

The condition implies that there exist sequences

$$r_n \to +\infty \quad \text{and} \quad c_n \to +\infty \tag{8}$$

such that for all n,

$$\|u\|_{C^1(B_{r_n})} \leq \frac{1}{2}\ln\ln r_n - c_n.$$

We may assume $\{\frac{1}{2}\ln\ln r_n - c_n\}$ is an increasing sequence. Pick an increasing function $c(r)$ such that, for all $r \geq 0$,

$$\|u\|_{C^1(B_r)} \leq c(r), \tag{9}$$

and for all n,

$$c(r_n) = \frac{1}{2}\ln\ln r_n - c_n. \tag{10}$$

For an arbitrary $L > 0$, we are going to estimate

$$T(L) = \sup\{t | X([0, t]) \text{ exists and } \|X\|_{C[0,t]} \leq L\}.$$

Note that along a geodesic $\gamma(t), e^{u(\gamma(t))}p(t)q(t) = \text{constant}$. Hence

$$p(t)q(t) = e^{u(\gamma(0))-u(\gamma(t))}p(0)q(0). \tag{11}$$

Denote

$$a(L) = e^{2c(L)}|p(0)q(0)|. \tag{12}$$

Then (9) and (11) show that for $t \in [0, T(L)]$,

$$|p(t)q(t)| \leq a(L),$$
$$p^2(t) + q^2(t) = p^2q^2(\alpha^2 + \beta^2) \leq a(L)^2\|X(t)\|^2, \tag{13}$$
$$|u_x(\gamma(t))| + |u_y(\gamma(t))| \leq 2c(L). \tag{14}$$

Since γ is the geodesic of $e^u dx\,dy$, therefore

$$X(t) = X_0 + \int_0^t (p(t), q(t), u_x(\gamma(t)), u_y(\gamma(t)))\,dt, \tag{15}$$

where $X_0 = (x(0), y(0), \alpha(0), \beta(0))$ is the initial data. From (13)–(15), we obtain that, for $t \in [0, T(L)]$,

$$\|X(t)\| \le \|X_0\| + \int_0^t (2c(L) + a(L)\|X(t)\|)\,dt. \tag{16}$$

Let

$$f(t) = \int_0^t \|X(t)\|dt; \tag{17}$$

then (16) is

$$f'(t) \le \|X_0\| + 2ct + af(t),$$

$$\frac{d}{dt}(e^{-at}f(t)) = e^{-at}(f' - af) \le e^{-at}(\|X_0\| + 2ct),$$

where $a = a(L)$ and $c = c(L)$. Integrating both sides, we get

$$e^{-at}f(t) \le \frac{2c}{a^2} + \frac{\|X_0\|}{a} - e^{-at}\left(\frac{2c}{a}t + \frac{2c}{a^2} + \frac{\|X_0\|}{a}\right),$$

$$f(t) \le \left(\frac{2c}{a^2} + \frac{\|X_0\|}{a}\right)(e^{at} - 1) - \frac{2c}{a}t. \tag{18}$$

Then (16)–(18) imply that, for $t \in [0, T(L)]$,

$$\|X(t)\| \le M_L(t), \tag{19}$$

where $M_L \in C^\infty(R)$ is defined as

$$M_L(t) = \|X_0\| + \left(\frac{2c(L)}{a(L)} + \|X_0\|\right)(e^{a(L)t} - 1).$$

Define

$$t_n = \frac{1}{a(r_n)} \ln \frac{a(r_n)r_n + 2c(r_n)}{a(r_n)\|X_0\| + 2c(r_n)}, \tag{20}$$

that is

$$M_{r_n}(t_n) = r_n.$$

Then (19) shows that, as long as $X(t)$ exists,

$$\|X(t)\| \le r_n \qquad \text{for } t \in [0, t_n]. \tag{21}$$

But (8),(10), (12) and (20) imply

$$t_n \to +\infty \qquad (n \to \infty). \tag{22}$$

Then (21) and (22) imply that there exists $L \in C(R)$ such that for all $t \ge 0$, as long as $X(t)$ exists,

$$\|X(t)\| \le L(t). \tag{23}$$

Now (13) and (23) imply a priori estimate

$$x(t)^2 + y(t)^2 + p(t)^2 + q(t)^2 \leq (1 + a(L(t))^2)L(t)^2,$$

which guarantees that $\gamma(t)$ exists for all $t \geq 0$. Q.E.D.

Remark 4. If $u \in C^1(R^2)$ satisfies (7) and has the following property:

> For any compact subset $K \subset R^2$, there exists $\varepsilon > 0$ such that for any initial data $p \in K, v \in S^1 = \{x^2 + y^2 = 1\} \subset R^2$, the geodesic $\gamma(t)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ exists unique on $t \in [0, \varepsilon]$;

then the Lorentzian metric $e^u dx\, dy$ on R^2 is complete. By Peano's existence theorem in the theory of ordinary differential equations, the local existence of the geodesic is always true. But the uniqueness might be false. If $u \in C^{1,\alpha}(R^2)$ $(0 < \alpha < 1)$, then the uniqueness is also true.

Theorem 5. *Let* $f \in C^k(R^2)$ $(1 \leq k \leq \infty)$,

$$F(r) = \sup_{(x,y)\in B_r} \int_{-r}^{r} (|f(x,t)| + |f(t,y)|)dt.$$

If there exist two nonnegative functions $g, h \in C(R)$ *and a constant* $c > 0$ *such that*

$$\|p\|_{L^1(R^2)} < \infty \qquad \text{where } p(x,y) = f(x,y)e^{-g(x)-h(y)},$$

and

$$\liminf_{r\to\infty R\geq r} (cF(R) + \|g\|_{C^1[-R,R]} + \|h\|_{C^1[-R,R]} - \frac{1}{2}\ln\ln R) = -\infty,$$

then f *is the curvature function for some complete* C^{k+1}*-Lorentzian metric* $e^u dx\, dy$ *on* R^2.

Proof. Pick $\varepsilon > 0$, such that

$$\|\varepsilon p\|_{L^1(R^2)} < \frac{1}{\pi},$$

and

$$\liminf_{r\to\infty R\geq r} (\varepsilon e F(R) + \|g\|_{C^1[-R,R]} + \|h\|_{C^1[-R,R]} - \frac{1}{2}\ln\ln R) = -\infty. \tag{24}$$

By Lemma 1, there exists $w \in C^{k+1}(R^2)$ satisfying

$$w(x,y) = -\int_0^x \int_0^y \varepsilon p(s,t)e^{w(s,t)} ds\, dt. \tag{25}$$

Clearly $\|w\|_{C(R^2)} \leq 1$. Then (25) implies, for all $r > 0$,

$$\|w\|_{C^1(B_r)} \leq 1 + \varepsilon e F(r).$$

Let

$$u(x,y) = w(x,y) - g(x) - h(y). \tag{26}$$

Then

$$\|u\|_{C^1(B_r)} \leq 1 + \|g\|_{C^1[-r,r]} + \|h\|_{C^1[-r,r]} + \varepsilon e F(r). \tag{27}$$

Then (24), (27) and theorem 4 imply that $e^u dx\,dy$ is a complete Lorentzian metric. (25) and (26) imply that $K(e^u dx\,dy) = 2\varepsilon f$. Then the complete metric $2\varepsilon e^u dx\,dy$ has the curvature f. Q.E.D.

Remark 5. Theorem 5 also holds for $f \in C^\alpha(R^2)$ $(0 < \alpha < 1)$.

Taking $g = h = 0$, we obtain the following:

Corollary 2. *Let* $f \in C^k(R^2)$ $(1 \leq k \leq \infty)$. *If* $\|f\|_{L^1(R^2)} < \infty$ *and if there exists a constant* c *such that*

$$\liminf_{r\to\infty} \inf_{R\geq r} \left(\sup_{(x,y)\in B_R} \int_{-R}^{R} (|f(x,t)| + |f(t,y)|)\,dt - c\ln\ln R \right) = -\infty,$$

then f *is the curvature function for some complete* C^{k+1}*-Lorentzian metric* $e^u dx\,dy$ *on* R^2.

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References

1. J. T. Burns, *Curvature functions on Lorentz 2-manifolds*, Pacific J. of Math. **70** (1977), 325–335.

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