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ON TWISTED FRÉCHET AND (LB)-SPACES

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ABSTRACT. We study twisted Fréchet spaces as well as twisted (LB)-spaces. We prove that a twisted space can have a nontwisted dual and that twisted spaces of a special class cannot be complemented in nontwisted spaces. We also give new examples of twisted spaces.

INTRODUCTION

Twisted spaces were introduced by the second author (in [16]) to answer a question about the structure of Fréchet spaces without a continuous norm; since then, they have been useful in many constructions involving Fréchet spaces as well as their duals in a variety of contexts. These constructions were used extensively (cf. [5]) to study spaces of linear mappings between locally convex spaces, by S. Dierolf and V. B. Moscatelli in [6], and by J. Bonet and S. Dierolf in [1]; also, the twisted construction appears in J. Taskinen's counterexample to Grothendieck's *Problème des Topologies* (cf. [17]), even if in a minor role.

For additional properties of these spaces, see references [4], [7], [8], [13], [15], and the detailed study by J. Bonet and S. Dierolf in [2] and [3].

Our intention here is to present some new examples of twisted spaces and some new properties. In particular, a modification of the method of [16] allows us to build nonreflexive twisted Fréchet spaces and to give examples of twisted spaces with nontwisted duals. The same example shows that a twisted Fréchet space and a countable product of Banach spaces can have the same dual. In the case of the twisted spaces of [16] (we call them "standard twisted spaces"), we can even prove that they are never complemented in a countable product of Fréchet spaces with a continuous norm.

Our notation is standard (cf. [10]). A *quojection* $E = \mathrm{quoj}_n(X_n, R_n)$ is the projective limit of a sequence of Banach spaces X_n and surjective maps R_n. A Fréchet space (resp., (DF)-space) is called *twisted* if it is not isomorphic to a countable product (resp., countable direct sum) of Fréchet spaces with continuous norms (resp., of (DF)-spaces with total bounded subsets) (cf. [15]).

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In particular, a quojection (resp., a strict (LB)-space) is twisted if and only if it is not isomorphic to a product (resp., direct sum) of Banach spaces.

If (X_n) is a sequence of Banach spaces and λ is a normal Banach sequence space, then $(\bigoplus_n X_n)_\lambda$ is the space of all sequences (x_n) such that $x_n \in X_n$ for all n and the sequence $(\|x_n\|)$ belongs to λ. This space is a Banach space for the obvious norm (cf. [2]). In the case when $X_k = X$ for $k \le n$ and $X_k = Y$ for $k > n$, we write $(\bigoplus_{k\le n} X \oplus (\bigoplus_{k>n} Y))_\lambda$.

1. Standard twisted spaces

The methods used in [15] and in [16] allow us to build twisted strict (LB)-spaces and so, by duality in the reflexive case, twisted quojections. We present a direct method of obtaining twisted quojections, which will be very useful later on.

Let X be a Banach space and Y a noncomplemented subspace of X. If λ is a normal Banach sequence space, for every n we put $Z_n = (\bigoplus_{k\le n} X \oplus (\bigoplus_{k>n} X/Y))_\lambda$. Let $R_n \colon Z_{n+1} \to Z_n$ be the surjective map which is the identity on every copy of X/Y and on the first n copies of X, and is the quotient map of X onto X/Y on the copy of X at the place $n+1$. If $E = \mathrm{quoj}_n(Z_n, R_n)$ we have

Theorem 1.1. *E is a twisted quojection.*

Proof. Suppose that E is not twisted; then E is a countable product of Banach spaces and it can be represented as $E = \mathrm{quoj}_n(H_n, S_n)$, where the H_n's are Banach spaces and for every $k < n$ $\ker S_{nk}$ is complemented in $H_n (S_{nk} = S_{k+1} \circ \cdots \circ S_{n-1} \circ S_n)$. Since Z_1 is a quotient of E, there exists a k such that Z_1 is a quotient of H_k, and for the same reason, there exists an n such that H_k is a quotient of Z_n. It is not difficult to see, using the fact that the sequences (Z_n, R_n) and (H_n, S_n) are equivalent, that

$$H_k = \left(\left(\bigoplus_{i\le n} X\right)\Big/ M \oplus \left(\bigoplus_{i>n} X/Y\right)\right)_\lambda$$

for some closed subspace M of $\oplus_{i\le n} X$.

Now let H_m be such that Z_{n+1} is a quotient of H_m. As before, there exists $j > n+2$ such that

$$H_m = \left(\bigoplus_{i\le n+1} X \oplus \left(\bigoplus_{i=n+2}^{j-1} X\right)\Big/ L \oplus \left(\bigoplus_{i\ge j} X/Y\right)\right)_\lambda,$$

with L a closed subspace of $\bigoplus_{i=n+2}^{j-1} X$. It is clear that $\ker S_{mk} = M \oplus Y \oplus L$. Since $\ker S_{mk}$ is complemented in H_m, the copy of Y in $\ker S_{mk}$ is complemented in H_m too; but this copy of Y is contained in a copy of X (at the place $n+1$) in H_m and so Y is complemented in X, which is false.

We conclude that E is a twisted quojection.

From now on, a quojection built with the method of Theorem 1.1, namely starting with a Banach space X, a noncomplemented subspace Y of X and a normal Banach sequence space λ, will be called a *standard twisted quojection.* Similarly, the twisted (LB)-spaces of [16], whose construction is dual to that of Theorem 1.1., will be called *standard twisted* (LB)*-spaces.*

Let us note some applications of Theorem 1.1. Let E be the standard twisted quojection built with $X = 1^{\infty}$, $Y = c_0$ and $\lambda = 1^2$. It is clear that $E'_\beta = \operatorname{ind}_n(Z'_n, R'_n)$ (β denotes the strong topology of E') and that

$$Z'_n = \left(\bigoplus_{k\le n} 1^{\infty'} \oplus \left(\bigoplus_{k>n} c_0^{\perp}\right)\right)_{1^2}.$$

Since $1^{\infty'} = 1^1 \oplus c_0^{\perp}$, it is easily seen that

$$E'_\beta = (1^1)^{(\mathbb{N})} \oplus \left(\bigoplus_n c_0^{\perp}\right)_{1^2};$$

hence E'_β is a direct sum of Banach spaces. Let us consider the Fréchet space $F = (c_0)^{\mathbb{N}}(\oplus_n 1^{\infty}/c_0)_{1^2}$. F is a product of Banach spaces, and obviously $F'_\beta \simeq E'_\beta$. We have proved the following:

Theorem 1.2. (a) *There exists standard twisted quojections whose duals are direct sums of Banach spaces and hence whose biduals are products of Banach spaces.*

(b) *There exists a standard twisted quojection and a product of Banach spaces whose duals are isomorphic.*

Remark 1.3. It is clear that the above example can be generalized by starting with any Banach space X not complemented in X''.

Remark 1.4. It is also possible to construct a standard twisted (LB)-space whose dual is not twisted. In fact, let X be a Banach space not complemented in X'' and define $W_n = (\bigoplus_{k\le n} X'' \oplus (\bigoplus_{k>n} X))_{1^2}$. If we put $G = \operatorname{ind}_n W_n$, then G is a twisted (LB)-space (cf. [16]); but it is easily seen (by using the decomposition $X''' = X' \oplus X^{\perp}$) that $G'_\beta = (X^{\perp})^{\mathbb{N}} \oplus (\bigoplus_n X')_{1^2}$ and hence that G'_β is a product of Banach spaces. In the case when $X = c_0$ we obtain

$$G'_\beta = ((1^{\infty}/c_0)')^{\mathbb{N}} \oplus \left(\bigoplus_n 1^1\right)_{1^2} = \left((1^{\infty}/c_0)^{(\mathbb{N})} \oplus \left(\bigoplus_n c_0\right)_{1^2}\right)',$$

which also shows that a twisted (LB)-space can have the same dual as a countable direct sum of Banach spaces.

We recall that it is not known if a twisted space can be complemented in a nontwisted one. The only known result up to now is the one in [14] about complemented subspaces of $(1^p)^{\mathbb{N}}$ $(1 \le p \le \infty)$, $(c_0)^{\mathbb{N}}$, and their duals. However, the following result shows that the answer to the above problem is negative for standard twisted spaces. Precisely, we have

Theorem 1.5. (a) *No standard twisted* (LB)*-space can be complemented in a countable direct sum of locally convex spaces with total bounded subsets.*

(b) *No standard twisted quojection can be complemented in a countable product of Fréchet spaces with continuous norms.*

Proof. We space prove (a) only; the proof of (b) is dual to that of (a) and similar to the proof of Theorem 1.1.

Let us suppose that G is a standard twisted (LB)-space which is complemented in a countable direct sum of spaces with a total bounded subset. Let F be the complement of G; then $F = \text{ind}_n F_n$, where each F_n is a closed subspace of F_{n+1} and has a total bounded set. (This latter follows from the fact that F is complemented in such a direct sum). Write G in the standard form $G = \text{ind}_n W_n$, where $W_n = (\bigoplus_{k\le n} X \oplus (\bigoplus_{k>n} Y))_\lambda$, Y not complemented in X. Then

$$F \oplus G = \text{ind}_n(F_n \oplus W_n) = \text{ind}_n \left(F_n \oplus \left(\left(\bigoplus_{k\le n} X \right) \oplus \left(\bigoplus_{k>n} Y \right) \right)_\lambda \right).$$

By hypothesis, $F\oplus G$ also has a representation as a direct sum $F\oplus G = \text{ind}_n H_n$, with H_n complemented in $F \oplus G$ and having a total bounded set. This latter assumption yields (up to subsequences) $F_n \oplus W_n \subset H_n \subset F_{n+1} \oplus W_{n+1}$ for all n. It follows that $H_n = (M_n \oplus (\bigoplus_{k>n+1} Y))_\lambda$, with M_n a closed subspace of $F_{n+1} \oplus (\bigoplus_{k\le n+1} X)$. Since H_n is complemented in $F \oplus G$, it is also complemented in $F_{n+2} \oplus W_{n+2} = (F_{n+2} \oplus ((\bigoplus_{k\le n+2} X) \oplus (\bigoplus_{k>n+2} Y))_\lambda)$; and then the first copy of Y in H_n is complemented in the copy of X in $F_{n+2} \oplus W_{n+2}$ at the place $n+2$. This is impossible because Y is not complemented in X, and the proof is complete.

Corollary 1.6. *A standard twisted Fréchet space cannot be complemented in any Fréchet space with an unconditional basis.*

Proof. Follows from Theorem 1.5(b) and the result in [8].

Remark 1.7. From the Theorem 1.5(a) and [7] it also follows that standard twisted (LB)-spaces cannot be complemented in any strict (LB)-space with an unconditional basis.

The last result of this section shows how it is possible, by starting from a standard twisted quojection, to build another twisted quojection which is a space of linear maps.

Let $E = \text{quoj}_n(Z_n, R_n)$ be a standard twisted quojection with $\lambda = 1^\infty$. Consider the space $L(1^1, E)$ of all continuous linear maps between 1^1 and E endowed with the strong topology, and let the map $S_n : L(1^1, Z_{n+1}) \to L(1^1, Z_n)$ be defined by $S_n(T) = R_n \circ T$. Since 1^1 has the lifting property and R_n is surjective, S_n is surjective too and this implies that

$$L(1^1, E) = \text{quoj}_n[L(1^1, Z_n), S_n].$$

But we have $Z_n = (\bigoplus_{k\leq n} X \oplus (\bigoplus_{k>n} X/Y))_{1^\infty}$ and hence

$$L(1^1, Z_n) = \left[\bigoplus_{k\leq n} L(1^1, X) \oplus \left(\bigoplus_{k>n} L(1^1, X/Y)\right)\right]_{1^\infty}.$$

Now the lifting property of 1^1 again implies that

$$L(1^1, X/Y) = L(1^1, X)/L(1^1, Y)$$

and that $L(1^1, Y)$ is not complemented in $L(1^1, X)$ since Y is not complemented in X. But then Theorem 1.1. applies, and we have

Proposition 1.8. *The space $L(1^1, E)$ is a standard twisted quojection.*

2. Other examples of twisted spaces

We now show a "singular" example of a twisted quojection, whose construction is different in spirit from that of [15] and [16].

Let p_n be positive real numbers such that $2 \leq p_1 < p_n < p_{n+1}$, and let Q: $L^{p_{n+1}}(0,1) \to L^{p_n}(0,1)$ be a quotient map (cf. [11]). If $E = \text{quoj}_n(L^{P_n}(0,1), Q_n)$, we have

Proposition 2.1. *Let X be an infinite-dimensional Banach subspace of E; then X is isomorphic to 1^2.*

Proof. All continuous seminorms of E are equivalent norms on X for large n; hence X is isomorphic to a subspace of $L^{p_n}(0,1)$ for large n. Since $p_n > 2$, the theorem of Kadec and Pełczynski on subspaces of $L^p(0,1)$ applies and shows that the only possibility is $X \simeq 1^2$ (cf. [12]).

Corollary 2.2. *Let F be a subspace of E. If F is a quojection, then F is either twisted or isomorphic to one of the following spaces: ω, 1^2, $1^2 \times \omega$, $(1^2)^{\mathbb{N}}$. In particular, E is twisted.*

We do not know examples of quojections without Banach subspaces; in this direction the space E is close to such an example since it contains only one Banach space, namely 1^2.

Now we note that the results of §1 hold also in the case of nuclear twisted spaces. We recall the standard construction in [16]: Let X be a nuclear (LB)-space with a total bounded set and let Y be a noncomplemented subspace of X which has a total bounded set too. If λ is a normal, nuclear (LB)-sequence-space different from φ, put $W_n = (\bigoplus_{k\leq n} X \oplus (\bigoplus_{k>n} Y))_\lambda$ and $G = \text{ind}_n W_n$. Then G'_β is a twisted, nuclear Fréchet space and we have the following result, which is the nuclear analogue of Theorem 1.5 with the same proof.

Theorem 2.3. *The space G'_β is never complemented in a countable product of Fréchet spaces with continuous norms. In particular, G'_β is not complemented in any nuclear Fréchet space with a basis.*

Finally, we conclude with a curious (and perhaps surprising) example of a twisted, nuclear (LF)-space (i.e., not isomorphic to a direct sum of Fréchet spaces) which is the strict inductive limit of copies of s, where s is the usual space of rapidly decreasing sequences.

Example 2.4. Let F be a closed, noncomplemented subspace of s such that $F \simeq s$ (see [18]) and put $W_n = (\bigoplus_{k\leq n} s \oplus (\bigoplus_{k>n} F))_s$. Then $W_n \simeq s$, and W_n is a closed subspace of W_{n+1}. Let $G = \operatorname{ind}_n W_n$; we prove that G is twisted. Suppose that G is isomorphic to a direct sum of Fréchet spaces F_n. If $H_n = \bigoplus_{k=1}^n F_k$, the well-known Grothendieck-factorization theorem ([9]) ensures that (up to subsequences) $W_n \subset H_n \subset W_{n+1}$ for all n. But then, as usual, the fact that H_n is complemented in H_{n+1} implies the complementation of F in s. Since this is not true, G must be twisted. Note that by [7], G cannot have a basis. Note also that the above construction can be set up with s replaced by any nuclear power series space of infinite type (cf. [16] and [18]).

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