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INDUCED CELLS

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Dedicated to Professor Cao Xihua on his 70th birthday

ABSTRACT. We define the concept of induced cells for affine Weyl groups which is compatible with the concept of induced unipotent classes under Lusztig's bijection between the set of two-sided cells of an affine Weyl group and the set of unipotent classes of a corresponding connected reductive algebraic group over $\mathbf{C}$.

Let G be a connected reductive algebraic group over $\mathbf{C}$ and T a maximal torus of G. The Weyl group $W_0 = N_G(T)/T$ of G then acts on the character group $P = \mathrm{Hom}(T, \mathbf{C})$ of T. Let $R \subset P$ be the root system of W_0. W_0 leaves stable the subgroup X of P generated by R. The semi-direct product $W = W_0 \ltimes X$ is an affine Weyl group. Let G' be a Levi subgroup of G containing T. Then the Weyl group $W_0' = N_{G'}(T)/T$ of G' is a parabolic subgroup of W_0 and leaves stable the subgroup X' of P generated by the corresponding root system $R' \subset R$. The semi-direct product $W' = W_0' \ltimes X'$ is also an affine Weyl group, which is not a parabolic subgroup of W although it is a subgroup of W.

Following Kazhdan and Lusztig (see [1]) we have the concept of two-sided cells of W', W. In this paper our main result is that for any two-sided cell c' of W' we define naturally a two-sided cell c of W and call it the induced cell of c' from W' to W (see Theorem 3.2). Recently, Lusztig established a bijection between the set of two-sided cells of W' (resp. W) and the set of unipotent classes of G' (resp. G) (see [4], note that G' is connected). Under the bijections the induced cells from W' to W are compatible with the induced unipotent classes from G' to G, which were introduced by Lusztig and Spaltenstein in [5].

1. ADMISSIBLE PAIRS

1.1 Let S, S', S_0, S_0' be the sets of simple reflections of W, W', W_0, W_0' respectively. For any subset I' of S' we denote by $W'^{I'}$ the parabolic

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subgroup of W' generated by the elements in I'. We similarly define W^I for any subset I of S. If $W'^{I'}$ is finite then there exists some subset I of S such that $W'^{I'}$ and W^I are isomorphic as Coxeter groups. We say that (I', I) is an admissible pair if there exists some $w \in W_0$ such that $wp(I)w^{-1} = p'(I')$, where $p\colon W \to W_0$, $p'\colon W' \to W'_0$ are the natural projections (note that W'_0 is a parabolic subgroup of W_0).

Proposition 1.2. *Let I' be a subset of S' such that $W'^{I'}$ is finite; then there exists some subset I of S such that (I', I) is admissible.*

Proof. It is no harm to assume that (W, S) is an irreducible affine Weyl group. For root systems in R we shall distinguish the systems of type D_2 from the systems of type $A_1 \times A_1$, type D_3 from A_3, also we shall distinguish a system of type A_1 from a system of type B_1 or C_1 if there exist roots in R of different lengths.

a. If $W'^{I'}$ is of type

$$A_{i_1} \times A_{i_2} \times \dots A_{i_m}$$

or of type

$$B_{i_1} \times A_{i_2} \times A_{i_3} \times \cdots \times A_{i_m}$$

or of type

$$C_{i_1} \times A_{i_2} \times A_{i_3} \times \cdots \times A_{i_m}$$

or of type

$$D_{i_1} \times A_{i_2} \times A_{i_3} \times \dots A_{i_m}$$

then we can find $w \in W'_0$ and $I'' \subset S'_0$ such that $wp'(I')w^{-1} = I''$. In this case we choose $I = I''$; then (I', I) is admissible.

b. Let W be of classical type and suppose that $W'^{I'}$ is of type

(1) $$B_{i_1} \times D_{i_2} \times A_{i_3} \times A_{i_4} \times \cdots \times A_{i_m}$$

or of type

(2) $$C_{i_1} \times C_{i_2} \times A_{i_3} \times A_{i_4} \times \cdots \times A_{i_m}$$

or of type

(3) $$D_{i_1} \times D_{i_2} \times A_{i_3} \times A_{i_4} \times \dots A_{i_m}$$

Then there exists some subset I_1 of S such that W^{I_1} is of type $B_{i_1} \times D_{i_2}$ or of type $C_{i_1} \times C_{i_2}$ or of type $D_{i_1} \times D_{i_2}$. Choose $I_2 \subset S - I_1$ such that $W^{I_1 \cup I_2}$ is of type (1) or of type (2) or of type (3). Through a detailed analysis of the root system R we can find $w \in W_0$ such that $wp(I_1 \cup I_2)w^{-1} = p'(I')$. Let $I = I_1 \cup I_2$, then (I', I) is admissable.

Thus the proposition is verified for classical types.

c. We now assume that W is of exceptional type. The case of $\widetilde{G}_2$ is trivial. Let W be of type $\widetilde{F}_4$. Thanks to a, we only need to consider the case of W'_0

being of type B_3 or C_3 and $W'^{I'}$ being of type D_3 or of type $C_1 \times C_2$. In these cases there exists a unique subset I of S such that W^I and $W'^{I'}$ are isomorphic as Coxeter groups and we can verify that (I', I) is admissible.

Now let W be of type $\widetilde{E}_8$. Again thanks to a, we only need to deal with the cases of W_0' being of type $D_4 \times A_1$, $D_4 \times A_2$, $D_5 \times A_1$, $D_5 \times A_2$, E_6, E_7, and $W'^{I'}$ being of type $D_{i_1} \times D_{i_2} \times A_{i_3} \times \cdots \times A_{i_m}$, E_6, E_7; then we can prove the proposition case by case. It is similar to deal with the case of W being of type $\widetilde{E}_6$ or $\widetilde{E}_7$.

The proposition is proved.

1.3. Let $\sigma: (W, S) \to (W, S)$ be an automorphism of (W, S) such that the restriction of σ on X coincides with the restriction of an inner automorphism of W. Then for any subset I of S with W^I finite there exists some $w \in W_0$ such that $wp(\sigma(I))w^{-1} = p(I)$.

2. Representations of Weyl groups

We state some results and constructions about the representations of Weyl groups which were found by Lusztig in [2].

2.1. For any irreducible representation E of $W_0 = H$ we can associate to it two polynomials $P_E(t)$, $\widetilde{P}_E(t)$ with rational coefficients in an indeterminate t as in [2]. Let $P_E(t) = \gamma_E t^{a_E} +$ higher power terms, $\widetilde{P}_E(t) = \widetilde{\gamma}_E t^{\tilde{a}_E} +$ higher power terms, $\gamma_E \neq 0$, $\widetilde{\gamma}_E \neq 0$. We denote by φ_H the set of (isomorphic classes of) irreducible representations E of H which satisfy the equality $a_E = \tilde{a}_E$.

2.2. Let $V = P \otimes \mathbf{C}$ and let H' be a subgroup of H. Then we have a direct sum decomposition $V = V' \oplus V^{H'}$ where VH' is the set of H'-invariant vectors of V and V' is a H'-invariant subspace of V. Let $P_i(V')$ be the space of homogeneous polynomials $V' \to \mathbf{C}$ of degree i. Let E' be an irreducible representation of H' which occurs in $P_a(V')$ with multiplicity 1 and doesn't occur in $P_i(V')$ if $i < a$. Then there exists a unique irreducible representation E of H which occurs in $\mathrm{Ind}_{H'}^H E'$ with multiplicity 1 and $a_E = a$; we denote it by $j_{H'}^H E'$. If H' is a Weyl group and $E' \in \varphi_{H'}$ then $j_{H'}^H E'$ exists.

2.3. Let Δ be the set of simple roots in $R = R_1 \times R_2 \times \cdots \times R_m$, where R_i are the irreducible components of R. Let α_i be the highest short root of R_i. For any subset $I \subset \Delta \cup \{-\alpha_i\}_{1 \leq q \leq m}$ with $I \not\supseteq \Delta_i \cup \{-\alpha_i\}$ for any $1 \leq i \leq m$ (where $\Delta_i = R_i \cap \Delta$), let H^I be the supgroup of H generated by reflections with respect to the roots in I. Let $\overline{\varphi}_H$ be the set of all irreducible representations of H (up to isomorphism) of the form $j_{H^I}^H E'$ for some $E' \in \varphi_{H^I}$.

For each unipotent element $u \in G$ Springer associated an irreducible representation of H (see [6]); tensor this representation with the sign representation of H and denote this tensor product by ρ_u. The map $u \to \rho_u$ gives rise to a bijection between the set of unipotent classes of G and $\overline{\varphi}_H$ (see [3, p. 345]).

2.4. It is known that there is a bijection between φ_{H^I} and the set of two-sided cells of H^I (where H^I is as in 2.3 and regarding it as a Coxeter group) (see [3, 5.25]). For any two-sided cell c_I of H^I we therefore get an irreducible representation $E(c_I) \in \varphi_{H^I}$ of H^I. Hence we can associate to c_I a unipotent class $O(c_I)$ of G such that for any $u \in O(c_I)$ we have $j_{H^I}^H E(c_I) = \rho_u$.

2.5. The above results and constructions also hold for $H' = W_0'$ and G'.

3. The induced cells

3.1. Let c' be a two-sided cell of W'. Lusztig showed that there exists some subset I' of S' such that $W'^{I'}$ is finite and $c' \cap W'^{I'} = c'_{I'} \neq \varnothing$ (see [4]). Let $I \subset S$ be such that (I', I) is admissible. Then for some $w \in W_0$ we have $wp(I)w^{-1} = p'(I')$ which gives rise to an isomorphism of Coxeter groups $i: W'^{I'} \to W^I$. Let $c_{i,I} = i(c'_{I'})$ and let c_i be the two-sided cell of W containing $c_{i,I}$.

Theorem 3.2. *c_i is independent of the choices of w, I', I and only depends on c'. We denote it by $\mathrm{Ind}_{W'}^{W} c'$ and call it the induced cell of c' from W' to W.*

3.3. Let G and G' be as in the beginning of the paper. We denote by $O(c)$ (resp. $O(c')$) the unipotent class of G (resp. G') corresponding to a two-sided cell c (resp. c') of W (resp. W') under the bijection established by Lusztig in [4, Th. 4.8]. For any subsets $I \subset S$, $I' \subset S'$ with W^I, $W'^{I'}$ finite, we define $O(c_I) = O(p(c_I))$ and $O(c'_{I'}) = O(p'(c'_{I'}))$, where $c_I = c \cap W^I$ and $c'_{I'} = c' \cap W'^{I'}$ (see 2.4, 2.5).

3.4. *Proof of Theorem* 3.2. Let I', I, i be as in 3.1. We prove the theorem by showing that $j_{p(W^I)}^{W_0} E(p(c_{i,I})) = \rho_u$, where $u \in G$ is an element in the induced unipotent class $\mathrm{Ind}_{G'}^{G} O(c')$ of $O(c')$ from G' to G (see [5]). Thus $O(c_i) = \mathrm{Ind}_{G'}^{G} O(c')$ is independent of the choices of w, I'. I and so is c_i.

We have $c' \cap W'^{I'} = c'_{I'} \neq \varnothing$, hence $O(c'_{I'}) = O(c')$ (see [3]). Let $v \in O(c')$, then $j_{p'(W'^{I'})}^{W_0'} E(p'(c'_{I'})) = \rho_v$ (2.5). Now (I', I) is admissible and $wp(I)w^{-1} = p'(I')$. From the definition of $j_{H'}^{H}$ in 2.2 and the properties of induced representations we see that

$$\begin{aligned} j_{p(W^I)}^{W_0} E(p(c_{i,I})) &= j_{p'(W'^{I'})}^{W_0} E(p'(c'_{I'})) \\ &= j_{W_0'}^{W_0} j_{p'(W'^{I'})}^{W_0'} E(p'(c'_{I'})) \\ &= j_{W_0'}^{W_0} \rho_v = \rho_u. \end{aligned}$$

The theorem is proved.

Corollary 3.5. *Let W_0'' be a parabolic subgroup of W_0', W'' be its affine Weyl group and c'' be a two-sided cell of W''; then $\mathrm{Ind}_{W''}^{W} c'' = \mathrm{Ind}_{W'}^{W} \mathrm{Ind}_{W''}^{W'} c''$.*

Corollary 3.6. *Let* c', $c, O(c')$, $O(c)$ *be as in 3.4. Then* $O(\mathrm{Ind}_{W'}^{W} c') = \mathrm{Ind}_{G'}^{G} O(c')$.

3.7. Using 1.3 and in the same way as in the proof of Theorem 3.2 we know that $\sigma(c) = c$ for any two-sided cell c of W, where σ is as in 1.3 (see [4]).

3.8. **Example**. Let (W, S) be of type $\widetilde{D}_5$ and (W', S') be of type $\widetilde{D}_4$, $S = \{s_1, s_2, s_3, s_4, s_0\}$ and $s_1 s_0 = s_0 s_1$, $s_4 s_5 = s_5 s_4$, $(s_1 s_2)^3 = (s_0 s_2)^3 = (s_2 s_3)^3 = (s_3 s_4)^3 = (s_3 s_5)^3 = 1$; $S' = \{s_2, s_3, s_4, s_5, t_0\}$ and $t_0 s_2 = s_2 t_0$, $(t_0 s_3)^3 = 1$. Let c_1, c_2, c_3 be two-sided cells of W' which contain $s_2 s_4$, $s_2 s_5, s_4 s_5$ respectively; each two of c_1, c_2, c_3 are then different. But $\mathrm{Ind}_{W'}^{W} c_1 = \mathrm{Ind}_{W'}^{W} c_2 = \mathrm{Ind}_{W'}^{W} c_3 = c$, the two-sided cell of W includes $\{s_1 s_0,\ s_4 s_5,\ s_1 s_5\}$. This shows that the induced cells of different cells W' from W' to W may be equal.

Let $\sigma: (W', S') \to (W', S')$ be such that $\sigma: s_2 \to s_2$, $s_3 \to s_3$, $s_4 \to s_5$, $s_5 \to s_4$, $t_0 \to t_0$; then $\sigma(c_1) = c_2 \neq c_1$ and in this case 3.7 is false.

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