PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 1, January 1990

CLASSIFICATION OF SKEW SYMMETRIC MATRICES

BERNDT BRENKEN

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. The group $\mathrm{GL}(d, \mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^d)$ acts on the $\mathbb{Z}$-module $\mathrm{Hom}(\Lambda^2\mathbb{Z}^d, \mathbb{Z}/a\mathbb{Z})$ by $\varphi \to \varphi(\alpha\wedge\alpha)$ $(\alpha \in \mathrm{Aut}\,\mathbb{Z}^d)$. Associated with each φ in $\mathrm{Hom}(\Lambda^2\mathbb{Z}^d, \mathbb{Z}/a\mathbb{Z})$ is a finite set of invariants completely describing the orbit of φ under this action. The result holds with $\mathbb{Z}$ replaced by an arbitrary commutative principal ideal domain.

NOTATION

In the following, R denotes a commutative principal ideal domain (characteristic not equal to 2), N the R-module $\bigoplus^d R = R^d$ with a fixed basis $\{e_i | i = 1, \ldots, d\}$ $(d \in \mathbb{N})$, and S a fixed complete set (containing 1) of nonassociates of R (S may be chosen multiplicatively closed). If A is a commutative ring, $m_d(A)$ denotes the ring of $d \times d$ matrices over A. If $a \in A$, the ideal aA is written (a) and π denotes the canonical quotient map $A \to A/(a)$. If, in addition, A is an R-algebra and $\varphi \in \mathrm{Hom}(\Lambda^2 N, A)$, write $\varphi = [a_1, \ldots, a_k]$ if $k \le d/2$, $(a_1) \supseteq (a_2) \supseteq \cdots \supseteq (a_k)$; $\varphi(e_{2i-1} \wedge e_{2i}) = a_i$ $(1 \le i \le k)$; and $\varphi(e_j \wedge e_h) = 0$ for all other $j < h$, $j, h \in \{1, \ldots, d\}$.

INTRODUCTION

Consider $\Phi \in m_d(A)$, A a commutative principal ideal ring. Let $(d_k(\Phi))$, the kth determinantal divisor of Φ, be the ideal of A generated by the determinants of the $k \times k$ submatrices of Φ $(1 \le k \le d)$. The greatest k with $(d_k(\Phi)) \neq 0$ is the rank of Φ. It is evident that $(d_k(\Phi)) \subseteq (d_{k-1}(\Phi))$ where $(d_0(\Phi))$ is defined to be A. For $k \ge 1$, let $(s_k(\Phi))$ denote the ideal $\{m \in A | md_{k-1}(\Phi) \in (d_k(\Phi)\}$ if $(d_k(\Phi)) \neq 0$, and zero otherwise. Then $(s_k(\Phi))(d_{k-1}(\Phi)) = (d_k(\Phi))$. Note that $(s_1(\Phi)) = (d_1(\Phi))$. Thus, for Φ_1, $\Phi_2 \in m_d(A)$, $(s_k(\Phi_1)) = (s_k(\Phi_2))$ for all k if and only if $(d_k(\Phi_1)) = (d_k(\Phi_2))$ for all k. If A is also an integral domain then $(s_k(\Phi)) \subseteq (s_{k-1}(\Phi))$ [3].

Received by the editors March 15, 1989 and, in revised form, April 24, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46K10, 46L40, 15A63, 46L80.
Key words and phrases. Module, principal ideal domain, pfaffian, skew symmetric matrix, automorphism.

Assume A is both a commutative R-algebra and a principal ideal ring. The group $G = \mathrm{Aut}(R^d)$ acts on the R-module $\mathrm{Hom}(\Lambda^2 R^d, A)$ by $\varphi \to \varphi(\alpha \wedge \alpha)$ ($\alpha \in \mathrm{Aut}(R^d)$). Each $\varphi \in \mathrm{Hom}(\Lambda^2 R^d, A)$ may be identified with a skew symmetric matrix $\Phi \in m_d(A)$ (the (i, j) entry of Φ is $\varphi(e_i \wedge e_j)$). For each $\alpha \in \mathrm{Aut}(R^d)$, i.e., for each invertible matrix α in $m_d(R)$, the element of $m_d(A)$ associated with $\varphi(\alpha \wedge \alpha)$ is $\alpha^t \Phi \alpha$ (where α^t is the transpose of α). Define $(d_k(\varphi))$ to be $(d_k(\Phi))$ and $(s_k(\varphi))$ to be $(s_k(\Phi))$. It follows that $(d_k(\varphi(\alpha \wedge \alpha))) = (d_k(\varphi))$ for all k and thus $(s_k(\varphi(\alpha \wedge \alpha))) = (s_k(\varphi))$ for all k [3].

Recall that two elements Φ_1, Φ_2 of $m_d(A)$ are congruent if there is an $\alpha \in \mathrm{Aut}(A^d)$ with $\Phi_1 = \alpha^t \Phi_2 \alpha$. They are equivalent if there are α, $\beta \in \mathrm{Aut}(A^d)$ with $\Phi_1 = \alpha \Phi_2 \beta$. Thus, if $A = R$ the G-orbit of φ in $\mathrm{Hom}(\Lambda^2 R^d, R)$ is the same as the congruence class of the skew symmetric matrix Φ. These congruence classes have been known for some time [3]. In this case, the rank r of φ is even, $(s_{2k-1}(\varphi)) = (s_{2k}(\varphi))$ and the sequence of ideals $(s_d(\varphi)) \subseteq (s_{d-1}(\varphi)) \subseteq \cdots \subseteq (s_1(\varphi))$ is a complete invariant for the orbit of φ in $\mathrm{Hom}(\Lambda^2 R^d, R)$. Furthermore, there is an $\alpha \in \mathrm{Aut}(R^d)$ with

$$\varphi(\alpha \wedge \alpha) = [s_2(\varphi), s_4(\varphi), \ldots, s_r(\varphi)].$$

It follows from this that elements equivalent in $\mathrm{Hom}(\Lambda^2 R^d, R)$ are also congruent [3].

The main new result below is a classification of the G-orbits in $\mathrm{Hom}(\Lambda^2 R^d, A)$ if A is the quotient, $R/(a)$, of R. An application is found in [1]. Although some special cases of the result (for example, when $R/(a)$ is a field, i.e., when (a) is prime.) are easy consequences of the classification result for $A = R$, the general case does not seem to be.

For $\Phi \in m_d(\mathbb{Z}/a\mathbb{Z})$ complete invariants for the class $\{\alpha\Phi\beta | \alpha, \beta \in \mathrm{SL}(d, \mathbb{Z})\}$ are given in [2]. These conditions coincide with those given below for the G-orbit of $\varphi \in \mathrm{Hom}(\Lambda^2 \mathbb{Z}^d, \mathbb{Z}/(a))$. We conclude that if Φ_1, Φ_2 are skew symmetric elements of $m_d(\mathbb{Z}/a\mathbb{Z})$ with $\Phi_1 = \alpha \Phi_2 \beta$ for some α, $\beta \in \mathrm{SL}(d, \mathbb{Z})$ then $\Phi_1 = \gamma^t \Phi_2 \gamma$ for some $\gamma \in \mathrm{SL}(d, \mathbb{Z})$.

Recall that if A is a commutative ring and $d = 2m$, then there is a polynomial Pf in $d(d-1)/2$ variables with coefficients in A called the generic pfaffian (of size d). It is homogeneous of degree m. If $\varphi \in \mathrm{Hom}(\Lambda^2 N, A)$ as before, write $\mathrm{Pf}(\varphi)$ for $Pf(\Phi)$, the value of Pf evaluated at the elements $\varphi(e_i \wedge e_j)$ $(i < j)$ of A. Note that $Pf(\varphi(\alpha \wedge \alpha)) = \mathrm{Pf}(\alpha^t \Phi \alpha) = (\det \alpha) \mathrm{Pf}(\Phi) = \det \alpha Pf(\varphi)$ for $\alpha \in \mathrm{Aut}(N)$.

Invariants of $\varphi \in \mathrm{Hom}(\Lambda^2 N, R/(a))$

Choose $a \in R$, $a \neq 0$, and let M denote the quotient $R/(a)$. In this section we describe invariants for the orbit of an element $\varphi \in \mathrm{Hom}(\Lambda^2 N, M)$ under the action of $\mathrm{Aut}(N)$.

For $\varphi \in \operatorname{Hom}(\Lambda^2 N, M)$, let $L(\varphi) = \{n \in N | \varphi(n \wedge g) = 0 \ (g \in N)\}$. Since $N \cong aN \subseteq L(\varphi) \subseteq N$, $L(\varphi)$ is a free submodule of N of rank d and $I(\varphi) = N/L(\varphi)$ is a finitely generated torsion module over R. For $\alpha \in \operatorname{Aut}(N)$ we have $\alpha^{-1}(L(\varphi)) = L(\varphi(\alpha\wedge\alpha))$; so α induces an isomorphism of $I(\varphi(\alpha\wedge\alpha))$ onto $I(\varphi)$. Thus the invariants of $I(\varphi)$, a sequence of decreasing ideals of R completely describing the isomorphism class of $I(\varphi)$, are also invariants of the orbit of φ.

To compute the invariants of $I(\varphi)$, choose $\psi \in \operatorname{Hom}(\Lambda^2 N, R)$ with $\pi\psi = \varphi$. Since there is an $\alpha \in \operatorname{Aut}(N)$ with $\psi(\alpha\wedge\alpha) = [s_2(\psi), s_4(\psi), \dots]$, it follows that $I(\varphi)$ has invariants $(p_d) \supseteq (p_{d-1}) \supseteq \cdots \supseteq (p_1)$, where p_j are chosen with $p_{2j-1} = p_{2j}$ and $(p_j)(a, s_j(\psi)) = (a)$. (Define $(p_{r+1}) = \cdots = (p_d) = (1)$ if the number of invariants of $I(\varphi)$ is r and r is less than d.) Since $(a) \subseteq (p_j)$ for all j, the ideals (p_j) of R may be viewed equally well as ideals of M. However, the ideals (p_j) have little relationship to the ideals $(s_j(\varphi))$ of M. Although $\pi(d_j(\psi)) = d_j(\varphi)$, we have $\pi(s_j(\psi)) \subseteq (s_j(\varphi))$ unless $(d_j(\varphi)) = 0$, in which case $0 = (s_j(\varphi)) \subseteq \pi(s_j(\psi))$.

Definition. For $\varphi \in \operatorname{Hom}(\Lambda^2 N, M)$ and $d = 2m$, let

$$C(\varphi) = \pi(Pf(\psi)\delta_1^{-1}\delta_2^{-1}\cdots\delta_{m-1}^{-1}).$$

Here $\psi \in \operatorname{Hom}(\Lambda^2 N, R)$ satisfies $\pi(\psi) = \varphi$, and $\delta_j \in S$ is such that $(\delta_j) = (s_{2j}(\psi), a)$, $j = 1, \dots, m$.

Of course, we must show that $C(\varphi)$ is well defined. Note first that the elements δ_j, $j = 1, \dots, m$, are uniquely determined by φ. This follows from the fact that the ideals $(a, s_j(\psi))$ of R are uniquely determined by the conditions $(p_j)(a, s_j(\psi)) = (a)$, where (p_j) are the invariants of $I(\varphi)$. Therefore, $(a, s_j(\psi)) = (a, s_j(\psi'))$ for ψ, $\psi' \in \operatorname{Hom}(\Lambda^2 N, R)$, with $\pi\psi = \pi\psi' = \varphi$. Choose $\alpha \in \operatorname{Aut}(N)$ with $\psi(\alpha\wedge\alpha) = [s_2(\psi), s_4(\psi), \dots, s_{2m}(\psi)]$. We have

$$\begin{aligned} s_2(\psi)\dots s_{2m}(\psi)\delta_1^{-1}\cdots\delta_{m-1}^{-1} &= Pf(\psi(\alpha\wedge\alpha))\delta_1^{-1}\cdots\delta_{m-1}^{-1} \\ &= \mu Pf(\psi)\delta_1^{-1}\cdots\delta_{m-1}^{-1}, \end{aligned}$$

where $\mu = \det\alpha$ is a unit of R. This also demonstrates that $\delta_1 \cdot \cdots \cdot \delta_{m-1}$ divides $Pf(\psi)$. Since $\psi'(\alpha\wedge\alpha) = \psi(\alpha\wedge\alpha) \bmod a$, $\psi'(\alpha\wedge\alpha)$ has entries in (a) or $s_j(\psi) + (a)$. Also, $\delta_j | s_k(\psi)$ $(k \geq 2j)$ and $\delta_j | a$. It follows from the form of the polynomial Pf that

$$\begin{aligned} \mu Pf(\psi')\delta_1^{-1}\cdots\delta_{m-1}^{-1} &= Pf(\psi'(\alpha\wedge\alpha))\delta_1^{-1}\cdots\delta_{m-1}^{-1} \\ &= \prod_{j=1}^{m}(s_{2j}(\psi) + x_j a)\delta_1^{-1}\cdots\delta_{m-1}^{-1} + \sum_t g_t\delta_1^{-1}\cdots\delta_{m-1}^{-1}, \end{aligned}$$

where $x_j \in R$ and $g_t \in R$ is divisible by $\delta_1\cdots\delta_{m-1}$ with $g_t\delta_1^{-1}\cdots\delta_{m-1}^{-1} \in (a)$.

The first term of the sum expands to

$$\prod_{j=1}^{m} s_{2j}(\psi)\delta_1^{-1}\cdots\delta_{m-1}^{-1} + a\sum_q f_q\delta_1^{-1}\cdots\delta_{m-1}^{-1}$$

with $f_q \in R$ divisible by $\delta_1\cdots\delta_{m-1}$. Therefore,

$$\mu Pf(\psi')\delta_1^{-1}\cdots\delta_{m-1}^{-1} = \mu Pf(\psi)\delta_1^{-1}\cdots\delta_{m-1}^{-1} + xa$$

for some $x \in R$, and

$$\pi(Pf(\psi')(\delta_1^{-1}\cdots\delta_{m-1}^{-1}) = \pi(Pf(\psi)\delta_1^{-1}\cdots\delta_{m-1}^{-1}).$$

Proposition. *If* $d = 2m$, $\varphi \in \mathrm{Hom}(\Lambda^2 N, M)$, *and* $\alpha \in \mathrm{Aut}(N)$, *then*

$$C(\varphi(\alpha \wedge \alpha)) = \pi(\det\alpha)C(\varphi).$$

Proof. Choose $\psi \in \mathrm{Hom}(\Lambda^2 N, R)$ with $\pi\psi = \varphi$ and $\delta_j \in S$ with $(\delta_j) = (s_{2j}(\psi), a)$, $j = 1, \ldots, m$. Since $I(\varphi(\alpha\wedge\alpha)) = I(\varphi)$, it follows that

$$s_{2j}(\psi(\alpha\wedge\alpha)), a) = (s_{2j}(\psi), a).$$

We have

$$\begin{aligned} C(\varphi(\alpha\wedge\alpha)) &= \pi(Pf(\psi(\alpha\wedge\alpha))\delta_1^{-1}\cdots\delta_{m-1}^{-1}) \\ &= \pi(\det(\alpha)Pf(\psi)\delta_1^{-1}\cdots\delta_{m-1}^{-1}) = \pi(\det\alpha)C(\varphi). \quad \square \end{aligned}$$

Note that $C(\varphi) = 0$ whenever $(\delta_j) = (a)$ for some j. Note also that $C(\varphi)$ is a finer invariant than $Pf(\varphi)$. For example, let R be the ring of integers, $d = 4$ and $a = 27$. If $\varphi_1 = [\pi(3), \pi(6)]$ and $\varphi_2 = [\pi(3), \pi(24)]$, then $I(\varphi_1) = I(\varphi_2)$ and $Pf(\varphi_1) = Pf(\varphi_2) = \pi(18)$. However, $C(\varphi_1) = \pi(6)$ and $C(\varphi_2) = \pi(24)$; so it follows from the result below that φ_1 and φ_2 are not in the same orbit under the action of $\mathrm{Aut}(\mathbb{Z}^4)$. These elements are congruent though, since there is an invertible $\alpha \in m_4(\mathbb{Z}/27)$ with $\alpha^t\varphi_1\alpha = \varphi_2$.

CLASSIFICATION OF ORBITS

Theorem 1. *Let* R *be a principal ideal domain,* $N = \bigoplus^d R$, $M = R/(a)$ *a quotient of* R, μ *a unit of* R, *and* φ_1, $\varphi_2 \in \mathrm{Hom}(\Lambda^2 N, M)$.

(i) *Assume* d *is odd. If there is an* $\alpha \in \mathrm{Aut}(N)$ *with* $\varphi_1(\alpha\wedge\alpha) \cong \varphi_2$, *then* $I(\varphi_1) \cong I(\varphi_2)$. *Conversely, if* $I(\varphi_1) \cong I(\varphi_2)$, *then there is an* $\alpha \in \mathrm{Aut}(N)$ *with* $\det\alpha = \mu$ *and* $\varphi_1(\alpha\wedge\alpha) = \varphi_2$.

(ii) *Assume* d *is even. There is an* $\alpha \in \mathrm{Aut}(N)$ *with* $\det\alpha = \mu$ *and* $\varphi_1(\alpha\wedge\alpha) = \varphi_2$ *if and only if* $I(\varphi_1) = I(\varphi_2)$ *and* $C(\varphi_2) = \pi(\mu)C(\varphi_1)$.

If $R/(a)$ is an integral domain, i.e., a field, then this theorem is easy. (Use the canonical form of a skew symmetric matrix over a field and the fact that for any $a \in R$, an element of $\mathrm{SL}(d, R/(a))$ can be lifted to an element of $\mathrm{SL}(d, R)$ [3].)

We first prove two Lemmas.

Lemma 1 (Use the preceding notation). *Let* $d = 3$, $\varphi_1 = [\pi(u)]$, $\varphi_2 = [\pi(v)]$, *and* $(u, a) = (v, a) = (\delta)$, *where* $u, v, \delta \in R$. *There is an* $\alpha \in \mathrm{Aut}(N)$ *with* $\det\alpha = \mu$ *and* $\varphi_1(\alpha \wedge \alpha) = \varphi_2$.

Proof. Choose $x, y \in R$ with $xu + ya = \delta$ and define $\varphi \in \mathrm{Hom}(\Lambda^2 N, M)$ by $\varphi = [\pi(\delta)]$. Using the fact that $\pi(a\delta^{-1})\pi(u) = \pi(a)\pi(u\delta^{-1}) = 0$ we have $\varphi_1(\gamma \wedge \gamma) = \varphi$, where $\gamma \in \mathrm{Aut}(N)$ is defined by $\gamma(e_1) = xe_1 + ye_3$, $\gamma(e_2) = e_2$, $\gamma(e_3) = -\mu a\delta^{-1}e_1 + \mu u\delta^{-1}e_3$. Note that $\det\gamma = \mu$. Similarly there is $\beta \in \mathrm{Aut}(N)$, with $\det\beta = 1$ and $\varphi_2(\beta \wedge \beta) = \varphi$. Let $\alpha = \gamma\beta^{-1}$. □

Lemma 2 (Use the preceding notation). *Let* $d = 4$, $\varphi_1 = [\pi(u_1), \pi(u_2)]$, $\varphi_2 = [\pi(v_1), \pi(v_2)]$, *and* $(\pi(u_i)) = (\pi(v_i))$, *where* u_i, $v_i \in R$ *for* $i = 1, 2$. *Assume* $\pi(\mu)C(\varphi_1) = C(\varphi_2)$. *Then there is an* $\alpha \in \mathrm{Aut}(N)$ *with* $\det\alpha = \mu$ *and* $\varphi_1(\alpha \wedge \alpha) = \varphi_2$.

Proof. Choose $\delta_i \in S$ with $(\delta_i) = (u_i, a) = \pi^{-1}(\pi(u_i))$, $i = 1, 2$. Then $\pi(\mu u_1 u_2 \delta_1^{-1}) = \pi(\mu)C(\varphi_1) = C(\varphi_2) = \pi(v_1 v_2 \delta_1^{-1})$. There are x, $y \in R$ with $xu_1 + ya = \delta_1$. Since $(v_1\delta_1^{-1}, a\delta_1^{-1}) = (1) = (x, a\delta_1^{-1})$ and since R is a unique factorization domain, we have $(z, (a\delta_1^{-1})^2) = (z, a\delta_1^{-1}) = (1)$, where $z = v_1\delta_1^{-1}x$. Thus there are p, $q \in R$ with $pz + q(a\delta_1^{-1})^2 = 1$. Define $\alpha \in \mathrm{Aut}(N)$ with $\det\alpha = \mu$ by $\alpha(e_1) = ze_1 + a\delta_1^{-1}e_3$, $\alpha(e_2) = e_2$, $\alpha(e_3) = -qa\delta_1^{-1}e_1 + pe_3$, and $\alpha(e_4) = \mu e_4$.

Since $\pi(z)\pi(u_1) = \pi(v_1\delta_1^{-1})\pi(\delta_1 - ya) = \pi(v_1)$, we have $\varphi_1(\alpha\wedge\alpha)(e_1 \wedge e_2) = \varphi_2(e_1 \wedge e_2)$. Note that

$$\begin{aligned}\pi(pzv_2) &= \pi(1 - q(a\delta_1^{-1})^2\pi(v_2) \\ &= \pi(v_2) - \pi(q(a\delta_1^{-1})(v_2\delta_1^{-1})a) \\ &= \pi(v_2)\end{aligned}$$

and

$$\pi(pzv_2) = \pi(pxv_1v_2\delta_1^{-1}).$$

By assumption this is $\pi(px\mu u_1u_2\delta_1^{-1})$ which is

$$\begin{aligned}\pi(p\mu u_2(u_1\delta_1^{-1})x) &= \pi(p\mu u_2(1 - ya\delta_1^{-1})) \\ &= \pi(p\mu u_2) - \pi(p\mu(u_2\delta_1^{-1})ya) \\ &= \pi(p\mu u_2).\end{aligned}$$

Thus, $\pi(v_2) = \pi(p\mu u_2)$ and $\varphi_1(\alpha\wedge\alpha)(e_3\wedge e_4) = \varphi_2(e_3\wedge e_4)$. It is straightforward to complete the check that $\varphi_1(\alpha \wedge \alpha) = \varphi_2$. □

The following fact is needed in the proof of Theorem 1(ii). If σ, $\rho \in R$ with $(\pi(\sigma)) = (\pi(\rho))$ (equivalently, $(\sigma, a) = (\rho, a)$), then there is a $w \in R$ with $\pi(w\sigma) = \pi(\rho)$ and $\pi(w)$ a unit in $R/(a)$. There is a straightforward proof of this using the Chinese Remainder theorem.

Proof of Theorem 1. (i) Let $d = 2m+1$. Choose $\psi_j \in \operatorname{Hom}(\Lambda^2 N, R)$ with $\pi\psi_j = \varphi_j$, $j = 1, 2$. Assuming $I(\varphi_1) \cong I(\varphi_2)$, we have $(a, s_k(\psi_1)) = (a, s_k(\psi_2))$ for $k = 1, \ldots, d$. There are $\beta_j \in \operatorname{Aut}(N)$ with $\varphi_j(\beta_j \wedge \beta_j) = [\pi(s_2(\psi_j)), \pi(s_4(\psi_j)), \ldots, \pi(s_{2m}(\psi_j))]$. Let N_1 denote the submodule of N spanned by $\{e_1, e_2, e_d\}$, N_r the submodule spanned by the remaining e_i, and I_r the identity map of N_r. Since $\Lambda^2 N = \Lambda^2 N_1 \oplus N_1 \otimes N_r \oplus \Lambda^2 N_r$, Lemma 1 ensures the existence of an $\alpha_1 \in \operatorname{Aut}(N)$ (of the form $\gamma_1 \oplus I_r$) with $\det(\alpha_1)$ any prescribed unit of R and

$$\varphi_1(\beta_1 \wedge \beta_1)(\alpha_1 \wedge \alpha_1) = [(\pi(s_2(\psi_2)), \pi(s_4(\psi_1)), \ldots].$$

Repeating this argument yields $\alpha_k \in \operatorname{Aut}(N)$, $k = 1, \ldots, m$, with

$$\varphi_1(\beta_1 \wedge \beta_1)(\alpha_1 \wedge \alpha_1) \cdots (\alpha_k \wedge \alpha_k) = \varphi_2(\beta_2 \wedge \beta_2).$$

By choosing $\det \alpha_k$ appropriately and setting $\alpha = \beta_1 \alpha_1 \cdots \alpha_m \beta_2^{-1}$, we have $\det \alpha = \mu$ and $\varphi_1(\alpha \wedge \alpha) = \varphi_2$.

(ii) Let $d = 2m$ and assume $I(\varphi_1) \cong I(\varphi_2)$ and $C(\varphi_2) = \pi(\mu) C(\varphi_1)$. If $\psi_j \in \operatorname{Hom}(\Lambda^2 N, R)$ are chosen with $\pi\psi_j = \varphi_j$, $j = 1, 2$, there are $\delta_i \in S$ with $(\delta_i) = (a, \sigma_i) = (a, \rho_i)$, where σ_i and ρ_i denote $s_{2i}(\psi_1)$ and $s_{2i}(\psi_2)$ respectively, $i = 1, \ldots, m$. Choose β_1, $\beta_2 \in \operatorname{Aut}(N)$ such that $\varphi_1(\beta_1 \wedge \beta_1) = [\pi(\sigma_1), \pi(\sigma_2), \ldots, \pi(\sigma_m)]$ and $\varphi_2(\beta_2 \wedge \beta_2) = [\pi(\rho_1), \pi(\rho_2), \ldots, \pi(\rho_m)]$. The condition $C(\varphi_2) = \pi(\mu) C(\varphi_1)$ is equivalent to

$$\pi(\rho_1 \cdots \rho_m \delta_1^{-1} \cdots \delta_{m-1}^{-1}) = \pi(\mu_2 \mu \mu_1^{-1}) \pi(\sigma_1 \cdots \sigma_m \delta_1^{-1} \cdots \delta_{m-1}^{-1}),$$

where $\mu_j = \det \beta_j$, $j = 1, 2$.

If $m = 1$, any $\gamma \in \operatorname{Aut}(N)$ with $\det \gamma = \mu_2 \mu \mu_1^{-1}$ satisfies $\varphi_1(\beta_1 \wedge \beta_1)(\gamma \wedge \gamma) = \varphi_2(\beta_2 \wedge \beta_2)$. If $\alpha = \beta_1 \gamma \beta_2^{-1}$, it follows that $\det \alpha = \mu$ and $\varphi_1(\alpha \wedge \alpha) = \varphi_2$. Henceforth assume $m > 1$.

Let x_i, $y_i \in R$ with $\rho_i x_i + a y_i = \delta_i$ and set $u_i = \sigma_i \delta_i^{-1} x_i$, $i = 1, \ldots, m-1$. Then

$$\pi(u_i \rho_i) = \pi(\sigma_i \delta_i^{-1}(\delta_i - a y_i)) = \pi(\sigma_i).$$

Also choose w_i, $v_i \in R$ with $\pi(w_i \rho_i) = \pi(\sigma_i)$ and $\pi(w_i v_i) = \pi(1)$, $i = 1, \ldots, m-1$. Then $\pi(v_i \sigma_i) = \pi(\rho_i)$ and

$$\pi(v_i u_i \rho_{m-1}) = \pi(v_i u_i \rho_i (\rho_i^{-1} \rho_{m-1})) = \pi(\rho_i \rho_i^{-1} \rho_{m-1}) = \pi(\rho_{m-1})$$

for $i \le m-1$.

For $i \le k \le m-1$ define $\xi_k \in \operatorname{Hom}(\Lambda^2 N, M)$ as

$$[\pi(\rho_1), \ldots, \pi(\rho_{k-1}), \pi(w_1 \cdots w_{k-1} \sigma_k), \pi(\sigma_{k+1}), \ldots, \pi(\sigma_m)].$$

Note that $\xi_1 = \varphi_1(\beta_1 \wedge \beta_1)$.

Since $\pi(w_1 \cdots w_{k-1} \sigma_k \sigma_{k+1} \delta_k^{-1}) = \pi(\rho_k w_1 \cdots w_k \sigma_{k+1} \delta_k^{-1})$ for $1 \le k \le m-2$, an application of Lemma 2 (recall that $\pi(w_i)$ are units of M) yields an $\alpha_k \in \operatorname{Aut}(N)$ with $\det \alpha_k = 1$ and $\xi_k(\alpha_k \wedge \alpha_k) = \xi_{k+1}$ $(1 \le k \le m-2)$.

We claim that

$$\pi(\mu_1^{-1}\mu\mu_2)\pi(w_1\cdots w_{m-2}\sigma_{m-1}\sigma_m\delta_{m-1}^{-1}) = \pi(\rho_{m-1}\rho_m\delta_{m-1}^{-1}),$$

from which it follows that there is an $\alpha_{m-1} \in \mathrm{Aut}(N)$ with $\det\alpha_{m-1} = \mu_1^{-1}\mu\mu_2$ and $\xi_{m-1}(\alpha_{m-1}\wedge\alpha_{m-1}) = \varphi_2(\beta_2\wedge\beta_2)$. We have

$$\begin{aligned}
&\pi(\mu_1^{-1}\mu\mu_2\delta_{m-1}^{-1}\sigma_m(w_1\cdots w_{m-2}\sigma_{m-1}))\\
&\quad= \pi(\mu_1^{-1}\mu\mu_2\delta_{m-1}^{-1}\sigma_m(w_1\cdots w_{m-2}u_{m-1}\rho_{m-1}))\\
&\quad= \pi(\mu_1^{-1}\mu\mu_2\delta_{m-1}^{-1}\sigma_m(w_1\cdots w_{m-2}(v_1u_1v_2u_2\cdots v_{m-2}u_{m-2}u_{m-1}\rho_{m-1})))\\
&\quad= \pi(\mu_1^{-1}\mu\mu_2\delta_{m-1}^{-1}\rho_{m-1}(w_1v_1)(w_2v_2)\cdots(w_{m-2}v_{m-2})u_1\cdots u_{m-1}\sigma_m)\\
&\quad= \pi(\mu_1^{-1}\mu\mu_2\delta_{m-1}^{-1}\rho_{m-1}\sigma_1\cdots\sigma_{m-1}x_1\cdots x_{m-1}\delta_1^{-1}\cdots\delta_{m-1}^{-1}\sigma_m)\\
&\quad= \pi(\delta_{m-1}^{-1}\rho_{m-1}\rho_1\cdots\rho_m\delta_1^{-1}\cdots\delta_{m-1}^{-1}x_1\cdots x_{m-1})\\
&\quad= \pi(\delta_{m-1}^{-1}\rho_{m-1}(\rho_1\delta_1^{-1}x_1)\cdots(\rho_{m-1}\delta_{m-1}^{-1}x_{m-1})\rho_m)\\
&\quad= \pi(\delta_{m-1}^{-1}\rho_{m-1}\rho_m).
\end{aligned}$$

The last equality follows from

$$\pi(\rho_i\delta_i^{-1}x_i\rho_m) = \pi((1-a\delta_i^{-1}y_i)\rho_m) = \pi(\rho_m) - \pi(\rho_m\delta_i^{-1}y_ia) = \pi(\rho_m).$$

Letting $\alpha = \beta_1\alpha_1\cdots\alpha_{m-1}\beta_2^{-1}$, we have $\det\alpha = \mu$ and $\varphi_1(\alpha\wedge\alpha) = \varphi_2$. □

Thus, if d is odd, the orbits in $\mathrm{Hom}(\Lambda^2 N, M)$ are the same for the action of $\mathrm{Aut}(N)$ and the action of the subgroup $\mathrm{SL}(d,R)$. Since an element of $\mathrm{SL}(d,M)$ can be lifted to an element of $\mathrm{SL}(d,R)$, we also know the orbit structure for the action of $\mathrm{SL}(d,M)$. If d is even, the orbit structure is now also clear for the action of $\mathrm{SL}(d,R)$ (and $\mathrm{SL}(d,M)$).

References

1. B. Brenken, *A classification of some noncommutative tori* (presented at the Great Plains Operator Theory Seminar, May, 1987 in Lawrence, Kansas, (to appear).
2. F. G. Frobenius, *Theorie der linearen Formen mit ganzen coefficienten*, J. Reine Angew. Math. **86** (1880), 96–116.
3. M. Newman, *Integral matrices*, Academic Press, 1972.

Mathematics Department, University of Calgary, Calgary, Alberta, T2N 1N4, Canada