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DEFICIENCY MODULES AND SPECIALIZATIONS

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ABSTRACT. Given a family of curves in the projective space we study how their deficiency modules can change. This has a geometrical translation in the problem of determining how the liaison class of a flat family of curves can change. As a consequence, we show that in every liaison class there are curves which are specializations of arithmetically Cohen–Macaulay curves.

1. INTRODUCTION AND PRELIMINARIES

The curves of the projective space $\mathbf{P}^3$ with fixed Hilbert polynomial are parameterized by the Hilbert scheme $H_{d,g}$. The cohomology dimensions $h^i(\mathbf{P}^3, I_{X_\alpha}(t))$, with $0 \leq i \leq 3$ and $t \in \mathbf{Z}$, when X_α varies in $H_{d,g}$, are upper semicontinuous and this is often used in order to distinguish between different irreducible components of $H_{d,g}$.

Here we investigate the possible variations of the module structures of the graded modules $\oplus_{t\in\mathbf{Z}} H^1(\mathbf{P}^3, I_{X_\alpha}(t))$. These modules, called Hartshorne–Rao modules or deficiency modules, have a very important geometrical meaning, since they individuate the liaison class of X_α.

In the second paragraph we prove that given any graded $k[x_0, x_1, x_2, x_3]$-module of finite length A and a quotient A/B, there exist irreducible families of curves whose general member is in the liaison class individuated by A/B, and which specialize to a curve in the liaison class individuated by A. As a corollary we get that every liaison class contains a curve which is specialization of a flat family of arithmetically Cohen–Macaulay curves. These families have the property that the dimensions $h^1(X_\alpha, O_{X_\alpha}(t))$ don't depend on α (we call them families with fixed speciality). This "quotient" condition is the natural one, since in the third paragraph we show that in a family of curves with fixed speciality the Hartshorne–Rao module of the special curve has a quotient which is in the closure (in a suitable variety) of the isomorphism class of the Hartshorne–Rao module of the general curve. Roughly speaking, the only obstruction is semicontinuity.

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If $\mathbf{k}$ is an algebraically closed field, we set $S = \mathbf{k}[x_0, x_1, x_2, x_3]$; with $\mathbf{P}^3$ we denote the three-dimensional projective space over $\mathbf{k}$; a *curve* will be a closed locally Cohen–Macaulay generically locally complete intersection one-dimensional subscheme of $\mathbf{P}^3$.

We need some generalities about liaison of curves and Hartshorne–Rao modules: for these we refer to the paper of Rao [R].

In [BB] we defined a variety parametrizing all possible structures of graded S-module which are compatible with a given "graded" $\mathbf{k}$-vector space structure. Let $M = \oplus_{i \in \mathbf{Z}} M_i$ be a graded S-module of finite length, and let $(m_1, \dots, m_t)$ be a t-ple of nonnegative integers such that $m_1 > 0$, $m_t > 0$. We say that M is of type $(m_1, \dots, m_t)$ if, up to shifting degrees, $M_i = 0$ if $i \leq 0$, $i > t$, and $\dim_{\mathbf{k}} M_i = m_i$ if $1 \leq i \leq t$ (every degree of M has a structure of k-vector space). Let now V be the vector space $V = \oplus_{i=1}^t \mathbf{k}^{m_i} = \oplus_{i=1}^t V_i$, and let us fix the canonical basis of V, and let $G = \{g \in S | \deg(g) = 1\} \cup \{0\}$.

Definition. $\mathscr{V} = \mathscr{V}_{\mathbf{P}^3}(m_1, \dots, m_t) = \{f = (f_1, \dots, f_{t-1}) \in \oplus_{i=1}^{t-1} \operatorname{Hom}(V_i \otimes G, V_{i+1}) | \forall g,\ h \in G,\ \forall \alpha \in V_i,\ \forall_i,\ 1 \leq i \leq t-2,\ f_{i+1}[f_i(\alpha \otimes g) \otimes h] = f_{i+1}[f_i(\alpha \otimes h) \otimes g]\}$ is called the *variety of module structures of finite length* over V of type $(m_1, \dots, m_t)$.

Fix a graded S-module $M = \oplus_{i=1}^t M_i$ of type $(m_1, \dots, m_t)$, and a basis $\mathscr{B}_i$ for the vector space M_i; then to $(M, \{\mathscr{B}_i\})$ is associated an element of $\mathscr{V}_{\mathbf{P}^3}(m_1, \dots, m_t)$ if we send each basis $\mathscr{B}_i$ to the canonical basis of V_i, since the multiplication is completely known if we know the vector space structure and the multiplication by elements of G. In this way we don't identify isomorphic module structure, but every isomorphism class of module structures is a locally closed irreducible subset of $\mathscr{V}$. The structure and the properties of $\mathscr{V}$ are described in [BB].

2. Existence of specializations

In this paragraph we prove that, under a very natural algebraic hypothesis on the Hartshorne–Rao modules, there exist families of curves belonging to a given liaison class which specialize to curves of a second liaison class, and that this specialization is with fixed speciality.

Definition. A flat family $p: Z \to W$ of curves in $\mathbf{P}^3$ (Z contained in $W \times \mathbf{P}^3$, and p induced by the projection) is said to have *fixed speciality* for all $t \in \mathbf{Z}$ and all $a \in W, b \in W$ the following equality is verified:

$$h^1(p^{-1}(a), O_{p^{-1}(a)}(t)) = h^1(p^{-1}(b), O_{p^{-1}(b)}(t)).$$

Lemma 2.1. *Let $\mathscr{F}$ and $\mathscr{G}$ be locally free sheaves, $rk\mathscr{F} \leq rk\mathscr{G} - 2$, and*

$$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G}$$

be a morphism which drops rank in codimension > 2. Then there exist a direct sum of line bundles $\mathscr{P}$, with $\operatorname{rank}\mathscr{P} = \operatorname{rank}\mathscr{G} - \operatorname{rank}\mathscr{F} - 1$ and a morphism

$0 \to \mathscr{P} \xrightarrow{\sigma} \mathscr{G}$ *such that*

$$(\varphi, \sigma) : \mathscr{F} \oplus \mathscr{P} \to \mathscr{G}$$

drops rank in codimension 2.

Proof. We have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{G}/\mathscr{F} \to 0.$$

Take $t >> 0$ such that $\mathscr{G}/\mathscr{F}(t)$ is globally generated and $H^1(\mathbf{P}^3, \mathscr{F}(t)) = 0$. Take $p = rk\mathscr{G} - rk\mathscr{F} - 1$ general sections of $\mathscr{G}/\mathscr{F}(t)$ and lift them to p sections of $\mathscr{G}(t)$. In such a way we get a morphism $0 \to (O(-t))^p \xrightarrow{\sigma} \mathscr{G}$ such that (φ, σ) drops rank in codimension 2.

Propositon 2.2. *Let A be a graded S-module of finite length, and A/B a quotient of A; let $\mathscr{L}$ and $\mathscr{M}$ be the liaison classes of curves of $\mathbf{P}^3$ whose Hartshorne–Rao modules are isomorphic to A and A/B respectively. Then there exist infinitely many (d, g) for which there is an irreducible family $\{X_t\}$ of curves of degree d and genus g with fixed speciality whose general member $X_t, t \neq t_0$, is in the liaison class $\mathscr{M}$, and $X_{t_0} \in \mathscr{L}$.*

Proof. We construct two locally free sheaves $\mathscr{A}$ and $\mathscr{B}$ whose dependency loci are curves in the liaison classes individuated by A and B, respectively, simply following Rao's construction. To do this, we consider a set of generators (as graded S-module) of B, and we extend it to a set of generators of A; we go on in constructing free resolutions of B and A in this way and we consider the second syzygies modules of B and A, let us call them B_2 and A_2. By sheafifying them we get two locally free sheaves $\mathscr{A}'$ and $\mathscr{B}'$ and, thanks to the functoriality of the sheafification, a morphism $\gamma: \mathscr{B}' \to \mathscr{A}'$. Exactly as in [R], one gets that

$$\oplus_t H^1(\mathbf{P}^3, \mathscr{B}'(t)) = B \qquad \text{and} \qquad \oplus_t H^1(\mathbf{P}^3, \mathscr{A}'(t)) = A,$$
$$H^2(\mathbf{P}^3, \mathscr{B}'(t)) = 0 = H^2(\mathbf{P}^3, \mathscr{A}'(t))$$

and the map induced by γ is exactly the inclusion $B \to A$.

By adding a suitable sum of line bundles $\mathscr{P}$ to $\mathscr{A}'$, we can find morphisms

$$\varphi = \gamma \oplus \sigma : \mathscr{B}' \to \mathscr{A}' \oplus \mathscr{P}$$

and

$$\varphi_0 = 0 \oplus \tau : \mathscr{B}' \to \mathscr{A}' \oplus \mathscr{P}$$

which are injective morphisms *of vector bundles*. Let us call $\mathscr{A}' \oplus \mathscr{P} = \mathscr{A}$.

Now we apply Lemma 2.1 to φ and to φ_0; there exist a direct sum of line bundles $\mathscr{H}$ and morphisms ψ, ψ_0 such that

$$\zeta: \quad (\varphi, \psi): \mathscr{B} \to \mathscr{A}$$

and

$$\mu: \quad (\varphi_0, \psi_0): \mathscr{B} \to \mathscr{A}$$

drop rank in codimension 2, where $\mathscr{B} = \mathscr{B}' \oplus \mathscr{H}$.

Morphisms having a two-codimensional dependency locus form an open nonvoid set in $\mathrm{Hom}(\mathscr{B}, \mathscr{A})$; hence there is an open nonvoid subset of $\mathbf{k}$ (the base field) for which the morphism $\Lambda_t = t\zeta + (1-t)\mu \colon \mathscr{B} \to \mathscr{A}$ drops rank in codimension two.

Thus we have, for these t, exact sequences

$$0 \to \mathscr{B} \xrightarrow{\Lambda_t} \mathscr{A} \to I_{X_t}(r) \to 0$$

and from the long exact sequences associated to them we see that $H^2(\mathbf{P}^3, I_{X_t}(p))$ does not depend on $t (H^2(\mathbf{P}^3, \mathscr{A}(t)) = 0)$. This shows that the family $\{X_t\}$ has fixed speciality.

Moreover, for $t \neq 0$ the map induced in cohomology by Λ_t is the inclusion $B \to A$; hence the Hartshorne–Rao module of X_t is isomorphic to A/B (since $H^2(\mathbf{P}^3, \mathscr{B}(t)) = 0$). For $t = 0$ the map induced in cohomology is zero, and hence the Hartshorne–Rao module of X_0 is isomorphic to A.

In order to get infinite values of (d, g) it is enough to use different twistings of the line bundles which appear in the construction.

Corollary 2.3. *Every liaison class contains a curve which is specialization of a flat family of arithmetically Cohen–Macaulay curves.*

Proof. Take $B = A$; the liaison class of curves with trivial Hartshorne–Rao module is exactly the class of arithmetically Cohen–Macaulay curves.

Example 2.4. We follow step by step the proof of the proposition of this section in order to produce an explicit example of this phenomenon.

Let $\mathbf{L}_1$ be the liaison class of two skew lines; the elements of this class have Hartshorne–Rao module concentrated in one degree, of dimension one. The vector bundle that arises from Rao's construction from this module is the cotangent bundle $\Omega_{\mathbf{P}^3}$. It is well known that there exists an exact sequence of vector bundles

$$0 \to \Omega_{\mathbf{P}^3} \xrightarrow{\tau} [O_{\mathbf{P}^3}(-1)]^4 \to O_{\mathbf{P}^3} \to 0.$$

Take as γ (Proposition 2.2) the identity map $1 \colon \Omega_{\mathbf{P}^3} \to \Omega_{\mathbf{P}^3}$. We have two injective morphisms of vector bundles

$$\Phi = 1 \oplus 0 \colon \quad \Omega_{\mathbf{P}^3} \to \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4,$$
$$\Phi_0 = 0 \oplus \tau \colon \quad \Omega_{\mathbf{P}^3} \to \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4,$$

whose cokernels are, respectively, $[O_{\mathbf{P}^3}(-1)]^4$ and $\Omega_{\mathbf{P}^3} \oplus O_{\mathbf{P}^3}$.

Since both $[O_{\mathbf{P}^3}(-1)]^4(2)$ and $\Omega_{\mathbf{P}^3} \oplus O_{\mathbf{P}^3}(2)$ are globally generated, and $H^1(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(2)) = 0$, we can find morphisms

$$\Psi, \Psi_0 \colon [O_{\mathbf{P}^3}(-2)]^3 \to \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4$$

such that

$$\zeta \colon (\Phi, \Psi) \colon \quad \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-2)]^3 \to \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4,$$
$$\mu \colon (\Phi_0, \Psi_0) \colon \quad \Omega_{\mathbf{P}^3} \oplus [\mathrm{O}_{\mathbf{P}^3}(-2)]^3 \to \Omega_{\mathbf{P}^3} \oplus [\mathrm{O}_{\mathbf{P}^3}(-1)]^4,$$

drop rank in codimension 2, and hence the family of morphisms Λ_t.

Therefore we have a family of exact sequences

$$0 \to \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-2)]^3 \xrightarrow{\Lambda_t} \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4 \to I_{X_t}(2) \to 0$$

(the twist of I_{X_t} is determined by the Chern classes), where X_t is a curve with $\deg(X_t) = 6$ and $g(X_t) = 3$ for every t.

Note that X_t is arithmetically Cohen–Macaulay for $t \neq 0$, whilst $X_0 \in \mathbf{L}_1$. Moreover,

$$H^0(\mathbf{P}^3, I_{X_t}(2)) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \neq 0. \end{cases}$$

Hence this is a family of hyperelliptic sextic curves (of genus 3), which specializes to a curve of bidegree $(2, 4)$ lying on a quadric surface (directly linked to two skew lines).

3. The Hartshorne–Rao module of a specialization

In this section we go in the opposite direction: that is to say, we study how we can change the Hartshorne–Rao module in a flat family of curves. In particular, we concentrate on the case of a family with fixed speciality.

Proposition 3.1. *Let $p: Z \to W$ be a flat family of smooth curves in $\mathbf{P}^3$, with $\dim(W) > 0$ and W irreducible, with fixed speciality. Assume the existence of $c \in W$ such that for all a, b, $\in W \setminus \{c\}$ the Hartshorne–Rao modules of $p^{-1}(a)$ and $p^{-1}(b)$ are isomorphic, and belong to the variety of module structures $\mathscr{V}$. Then the Hartshorne–Rao module M of $p^{-1}(c)$ has a quotient $Q \in \mathscr{V}$ such that Q is in the closure in $\mathscr{V}$ of the set of modules isomorphic to the Hartshorne–Rao module of $p^{-1}(a)$.*

Proof. Without losing generality we may assume W reduced, affine and of dimension 1. By base-change theorem ([M], Cor. 2 p. 50–51) for all $t \in \mathbf{Z}$ $R^1 p * O_Z(t)$ (and $R^2 p * I_Z(t)$) are locally free and $R^1 p * I_Z(t)$ is locally free on $W \setminus \{c\}$. Let T_t be the torsion part of $R^1 p * I_Z(t)$: T_t is concentrated in t (or it is even zero). Set $F_t = (R^1 p * I_Z(t))/T_t$. Since W is a smooth curve, F_1 is locally free. Let W' be a neighborhood of c such that over W' all F_t are free: it exists since there is only a finite number of nonzero F_t's. Fix a basis of F_t on W' for all t: this allows to define a morphism $j: W' \to \mathscr{V}$ such that if a, $b \in W' \setminus \{c\}$, then $j(a)$ is isomorphic to $j(b)$. Let us denote by $j_t(a)$ the tth graded component of $j(a)$. Thus $j(c)$ is in the closure of the orbit of $j(b), b \in W' \setminus \{c\}$. Again by base-change theorem, the natural maps

$$R^1 p * I_Z(t) \otimes k(c) \to H^1(\mathbf{P}^3, I_{p^{-1}(c)}(t))$$

are isomorphisms for every t.

Since tensor product is right exact, $j_t(c) = F_t \otimes k(c)$ for every t is a quotient (as vector space) of $H^1(\mathbf{P}^3, I_{p^{-1}(c)}(t))$, and since these maps are natural and commute with the multiplication maps we get that $j(c)$ is isomorphic, as graded module, to a quotient of $\oplus_t H^1(\mathbf{P}^3, I_{p^{-1}(c)}(t))$, that is to say to a quotient of the Hartshorne–Rao module of $p^{-1}(c)$.

Note that we can apply the base-change theorem since we are dealing with a family of curves with fixed speciality.

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