

## DEFICIENCY MODULES AND SPECIALIZATIONS

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**ABSTRACT.** Given a family of curves in the projective space we study how their deficiency modules can change. This has a geometrical translation in the problem of determining how the liaison class of a flat family of curves can change. As a consequence, we show that in every liaison class there are curves which are specializations of arithmetically Cohen–Macaulay curves.

### 1. INTRODUCTION AND PRELIMINARIES

The curves of the projective space  $\mathbf{P}^3$  with fixed Hilbert polynomial are parameterized by the Hilbert scheme  $H_{d,g}$ . The cohomology dimensions  $h^i(\mathbf{P}^3, I_{X_\alpha}(t))$ , with  $0 \leq i \leq 3$  and  $t \in \mathbf{Z}$ , when  $X_\alpha$  varies in  $H_{d,g}$ , are upper semicontinuous and this is often used in order to distinguish between different irreducible components of  $H_{d,g}$ .

Here we investigate the possible variations of the module structures of the graded modules  $\oplus_{t \in \mathbf{Z}} H^1(\mathbf{P}^3, I_{X_\alpha}(t))$ . These modules, called Hartshorne–Rao modules or deficiency modules, have a very important geometrical meaning, since they individuate the liaison class of  $X_\alpha$ .

In the second paragraph we prove that given any graded  $k[x_0, x_1, x_2, x_3]$ -module of finite length  $A$  and a quotient  $A/B$ , there exist irreducible families of curves whose general member is in the liaison class individuated by  $A/B$ , and which specialize to a curve in the liaison class individuated by  $A$ . As a corollary we get that every liaison class contains a curve which is specialization of a flat family of arithmetically Cohen–Macaulay curves. These families have the property that the dimensions  $h^1(X_\alpha, O_{X_\alpha}(t))$  don't depend on  $\alpha$  (we call them families with fixed speciality). This “quotient” condition is the natural one, since in the third paragraph we show that in a family of curves with fixed speciality the Hartshorne–Rao module of the special curve has a quotient which is in the closure (in a suitable variety) of the isomorphism class of the Hartshorne–Rao module of the general curve. Roughly speaking, the only obstruction is semicontinuity.

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If  $\mathbf{k}$  is an algebraically closed field, we set  $S = \mathbf{k}[x_0, x_1, x_2, x_3]$ ; with  $\mathbf{P}^3$  we denote the three-dimensional projective space over  $\mathbf{k}$ ; a *curve* will be a closed locally Cohen–Macaulay generically locally complete intersection one-dimensional subscheme of  $\mathbf{P}^3$ .

We need some generalities about liaison of curves and Hartshorne–Rao modules: for these we refer to the paper of Rao [R].

In [BB] we defined a variety parametrizing all possible structures of graded  $S$ -module which are compatible with a given “graded”  $\mathbf{k}$ -vector space structure. Let  $M = \bigoplus_{i \in \mathbf{Z}} M_i$  be a graded  $S$ -module of finite length, and let  $(m_1, \dots, m_t)$  be a  $t$ -ple of nonnegative integers such that  $m_1 > 0, \dots, m_t > 0$ . We say that  $M$  is of type  $(m_1, \dots, m_t)$  if, up to shifting degrees,  $M_i = 0$  if  $i \leq 0, i > t$ , and  $\dim_{\mathbf{k}} M_i = m_i$  if  $1 \leq i \leq t$  (every degree of  $M$  has a structure of  $\mathbf{k}$ -vector space). Let now  $V$  be the vector space  $V = \bigoplus_{i=1}^t \mathbf{k}^{m_i} = \bigoplus_{i=1}^t V_i$ , and let us fix the canonical basis of  $V$ , and let  $G = \{g \in S \mid \deg(g) = 1\} \cup \{0\}$ .

**Definition.**  $\mathcal{V} = \mathcal{V}_{\mathbf{P}^3}(m_1, \dots, m_t) = \{f = (f_1, \dots, f_{t-1}) \in \bigoplus_{i=1}^{t-1} \text{Hom}(V_i \otimes G, V_{i+1}) \mid \forall g, h \in G, \forall \alpha \in V_i, \forall_i, 1 \leq i \leq t-2, f_{i+1}[f_i(\alpha \otimes g) \otimes h] = f_{i+1}[f_i(\alpha \otimes h) \otimes g]\}$  is called the *variety of module structures of finite length* over  $V$  of type  $(m_1, \dots, m_t)$ .

Fix a graded  $S$ -module  $M = \bigoplus_{i=1}^t M_i$  of type  $(m_1, \dots, m_t)$ , and a basis  $\mathcal{B}_i$  for the vector space  $M_i$ ; then to  $(M, \{\mathcal{B}_i\})$  is associated an element of  $\mathcal{V}_{\mathbf{P}^3}(m_1, \dots, m_t)$  if we send each basis  $\mathcal{B}_i$  to the canonical basis of  $V_i$ , since the multiplication is completely known if we know the vector space structure and the multiplication by elements of  $G$ . In this way we don’t identify isomorphic module structure, but every isomorphism class of module structures is a locally closed irreducible subset of  $\mathcal{V}$ . The structure and the properties of  $\mathcal{V}$  are described in [BB].

## 2. EXISTENCE OF SPECIALIZATIONS

In this paragraph we prove that, under a very natural algebraic hypothesis on the Hartshorne–Rao modules, there exist families of curves belonging to a given liaison class which specialize to curves of a second liaison class, and that this specialization is with fixed speciality.

**Definition.** A flat family  $p: Z \rightarrow W$  of curves in  $\mathbf{P}^3$  ( $Z$  contained in  $W \times \mathbf{P}^3$ , and  $p$  induced by the projection) is said to have *fixed speciality* for all  $t \in \mathbf{Z}$  and all  $a \in W, b \in W$  the following equality is verified:

$$h^1(p^{-1}(a), \mathcal{O}_{p^{-1}(a)}(t)) = h^1(p^{-1}(b), \mathcal{O}_{p^{-1}(b)}(t)).$$

**Lemma 2.1.** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be locally free sheaves,  $\text{rk}\mathcal{F} \leq \text{rk}\mathcal{G} - 2$ , and*

$$0 \rightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{G}$$

*be a morphism which drops rank in codimension  $> 2$ . Then there exist a direct sum of line bundles  $\mathcal{P}$ , with  $\text{rank}\mathcal{P} = \text{rank}\mathcal{G} - \text{rank}\mathcal{F} - 1$  and a morphism*

$0 \rightarrow \mathcal{P} \xrightarrow{\sigma} \mathcal{G}$  such that

$$(\varphi, \sigma) : \mathcal{F} \oplus \mathcal{P} \rightarrow \mathcal{G}$$

drops rank in codimension 2.

*Proof.* We have an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{F} \rightarrow 0.$$

Take  $t \gg 0$  such that  $\mathcal{G}/\mathcal{F}(t)$  is globally generated and  $H^1(\mathbf{P}^3, \mathcal{F}(t)) = 0$ . Take  $p = rk\mathcal{G} - rk\mathcal{F} - 1$  general sections of  $\mathcal{G}/\mathcal{F}(t)$  and lift them to  $p$  sections of  $\mathcal{G}(t)$ . In such a way we get a morphism  $0 \rightarrow (O(-t))^p \xrightarrow{\sigma} \mathcal{G}$  such that  $(\varphi, \sigma)$  drops rank in codimension 2.

**Propositon 2.2.** *Let  $A$  be a graded  $S$ -module of finite length, and  $A/B$  a quotient of  $A$ ; let  $\mathcal{L}$  and  $\mathcal{M}$  be the liaison classes of curves of  $\mathbf{P}^3$  whose Hartshorne–Rao modules are isomorphic to  $A$  and  $A/B$  respectively. Then there exist infinitely many  $(d, g)$  for which there is an irreducible family  $\{X_t\}$  of curves of degree  $d$  and genus  $g$  with fixed speciality whose general member  $X_t, t \neq t_0$ , is in the liaison class  $\mathcal{M}$ , and  $X_{t_0} \in \mathcal{L}$ .*

*Proof.* We construct two locally free sheaves  $\mathcal{A}$  and  $\mathcal{B}$  whose dependency loci are curves in the liaison classes individuated by  $A$  and  $B$ , respectively, simply following Rao’s construction. To do this, we consider a set of generators (as graded  $S$ -module) of  $B$ , and we extend it to a set of generators of  $A$ ; we go on in constructing free resolutions of  $B$  and  $A$  in this way and we consider the second syzygies modules of  $B$  and  $A$ , let us call them  $B_2$  and  $A_2$ . By sheafifying them we get two locally free sheaves  $\mathcal{A}'$  and  $\mathcal{B}'$  and, thanks to the functoriality of the sheafification, a morphism  $\gamma : \mathcal{B}' \rightarrow \mathcal{A}'$ . Exactly as in [R], one gets that

$$\begin{aligned} \oplus_t H^1(\mathbf{P}^3, \mathcal{B}'(t)) = B \quad \text{and} \quad \oplus_t H^1(\mathbf{P}^3, \mathcal{A}'(t)) = A, \\ H^2(\mathbf{P}^3, \mathcal{B}'(t)) = 0 = H^2(\mathbf{P}^3, \mathcal{A}'(t)) \end{aligned}$$

and the map induced by  $\gamma$  is exactly the inclusion  $B \rightarrow A$ .

By adding a suitable sum of line bundles  $\mathcal{P}$  to  $\mathcal{A}'$ , we can find morphisms

$$\varphi = \gamma \oplus \sigma : \mathcal{B}' \rightarrow \mathcal{A}' \oplus \mathcal{P}$$

and

$$\varphi_0 = 0 \oplus \tau : \mathcal{B}' \rightarrow \mathcal{A}' \oplus \mathcal{P}$$

which are injective morphisms of vector bundles. Let us call  $\mathcal{A}' \oplus \mathcal{P} = \mathcal{A}$ .

Now we apply Lemma 2.1 to  $\varphi$  and to  $\varphi_0$ ; there exist a direct sum of line bundles  $\mathcal{H}$  and morphisms  $\psi, \psi_0$  such that

$$\zeta : (\varphi, \psi) : \mathcal{B} \rightarrow \mathcal{A}$$

and

$$\mu : (\varphi_0, \psi_0) : \mathcal{B} \rightarrow \mathcal{A}$$

drop rank in codimension 2, where  $\mathcal{B} = \mathcal{B}' \oplus \mathcal{H}$ .

Morphisms having a two-codimensional dependency locus form an open non-void set in  $\text{Hom}(\mathcal{B}, \mathcal{A})$ ; hence there is an open nonvoid subset of  $\mathbf{k}$  (the base field) for which the morphism  $\Lambda_t = t\zeta + (1 - t)\mu: \mathcal{B} \rightarrow \mathcal{A}$  drops rank in codimension two.

Thus we have, for these  $t$ , exact sequences

$$0 \rightarrow \mathcal{B} \xrightarrow{\Lambda_t} \mathcal{A} \rightarrow I_{X_t}(r) \rightarrow 0$$

and from the long exact sequences associated to them we see that  $H^2(\mathbf{P}^3, I_{X_t}(p))$  does not depend on  $t(H^2(\mathbf{P}^3, \mathcal{A}(t)) = 0)$ . This shows that the family  $\{X_t\}$  has fixed speciality.

Moreover, for  $t \neq 0$  the map induced in cohomology by  $\Lambda_t$  is the inclusion  $B \rightarrow A$ ; hence the Hartshorne–Rao module of  $X_t$  is isomorphic to  $A/B$  (since  $H^2(\mathbf{P}^3, \mathcal{B}(t)) = 0$ ). For  $t = 0$  the map induced in cohomology is zero, and hence the Hartshorne–Rao module of  $X_0$  is isomorphic to  $A$ .

In order to get infinite values of  $(d, g)$  it is enough to use different twistings of the line bundles which appear in the construction.

**Corollary 2.3.** *Every liaison class contains a curve which is specialization of a flat family of arithmetically Cohen–Macaulay curves.*

*Proof.* Take  $B = A$ ; the liaison class of curves with trivial Hartshorne–Rao module is exactly the class of arithmetically Cohen–Macaulay curves.

**Example 2.4.** We follow step by step the proof of the proposition of this section in order to produce an explicit example of this phenomenon.

Let  $L_1$  be the liaison class of two skew lines; the elements of this class have Hartshorne–Rao module concentrated in one degree, of dimension one. The vector bundle that arises from Rao’s construction from this module is the cotangent bundle  $\Omega_{\mathbf{P}^3}$ . It is well known that there exists an exact sequence of vector bundles

$$0 \rightarrow \Omega_{\mathbf{P}^3} \xrightarrow{\tau} [O_{\mathbf{P}^3}(-1)]^4 \rightarrow O_{\mathbf{P}^3} \rightarrow 0.$$

Take as  $\gamma$  (Proposition 2.2) the identity map  $1: \Omega_{\mathbf{P}^3} \rightarrow \Omega_{\mathbf{P}^3}$ . We have two injective morphisms of vector bundles

$$\begin{aligned} \Phi = 1 \oplus 0: \quad \Omega_{\mathbf{P}^3} &\rightarrow \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4, \\ \Phi_0 = 0 \oplus \tau: \quad \Omega_{\mathbf{P}^3} &\rightarrow \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4, \end{aligned}$$

whose cokernels are, respectively,  $[O_{\mathbf{P}^3}(-1)]^4$  and  $\Omega_{\mathbf{P}^3} \oplus O_{\mathbf{P}^3}$ .

Since both  $[O_{\mathbf{P}^3}(-1)]^4(2)$  and  $\Omega_{\mathbf{P}^3} \oplus O_{\mathbf{P}^3}(2)$  are globally generated, and  $H^1(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(2)) = 0$ , we can find morphisms

$$\Psi, \Psi_0: [O_{\mathbf{P}^3}(-2)]^3 \rightarrow \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4$$

such that

$$\begin{aligned} \zeta: (\Phi, \Psi): \quad \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-2)]^3 &\rightarrow \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4, \\ \mu: (\Phi_0, \Psi_0): \quad \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-2)]^3 &\rightarrow \Omega_{\mathbf{P}^3} \oplus [O_{\mathbf{P}^3}(-1)]^4, \end{aligned}$$

drop rank in codimension 2, and hence the family of morphisms  $\Lambda_t$ .

Therefore we have a family of exact sequences

$$0 \rightarrow \Omega_{\mathbf{P}^3} \oplus [\mathcal{O}_{\mathbf{P}^3}(-2)]^3 \xrightarrow{\Lambda_t} \Omega_{\mathbf{P}^3} \oplus [\mathcal{O}_{\mathbf{P}^3}(-1)]^4 \rightarrow I_{X_t}(2) \rightarrow 0$$

(the twist of  $I_{X_t}$  is determined by the Chern classes), where  $X_t$  is a curve with  $\text{deg}(X_t) = 6$  and  $g(X_t) = 3$  for every  $t$ .

Note that  $X_t$  is arithmetically Cohen–Macaulay for  $t \neq 0$ , whilst  $X_0 \in \mathbf{L}_1$ . Moreover,

$$H^0(\mathbf{P}^3, I_{X_t}(2)) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \neq 0. \end{cases}$$

Hence this is a family of hyperelliptic sextic curves (of genus 3), which specializes to a curve of bidegree  $(2, 4)$  lying on a quadric surface (directly linked to two skew lines).

### 3. THE HARTSHORNE–RAO MODULE OF A SPECIALIZATION

In this section we go in the opposite direction: that is to say, we study how we can change the Hartshorne–Rao module in a flat family of curves. In particular, we concentrate on the case of a family with fixed speciality.

**Proposition 3.1.** *Let  $p: Z \rightarrow W$  be a flat family of smooth curves in  $\mathbf{P}^3$ , with  $\dim(W) > 0$  and  $W$  irreducible, with fixed speciality. Assume the existence of  $c \in W$  such that for all  $a, b, \in W \setminus \{c\}$  the Hartshorne–Rao modules of  $p^{-1}(a)$  and  $p^{-1}(b)$  are isomorphic, and belong to the variety of module structures  $\mathcal{V}$ . Then the Hartshorne–Rao module  $M$  of  $p^{-1}(c)$  has a quotient  $Q \in \mathcal{V}$  such that  $Q$  is in the closure in  $\mathcal{V}$  of the set of modules isomorphic to the Hartshorne–Rao module of  $p^{-1}(a)$ .*

*Proof.* Without losing generality we may assume  $W$  reduced, affine and of dimension 1. By base-change theorem ([M], Cor. 2 p. 50–51) for all  $t \in \mathbf{Z}$   $R^1 p_* \mathcal{O}_Z(t)$  (and  $R^2 p_* \mathcal{I}_Z(t)$ ) are locally free and  $R^1 p_* \mathcal{I}_Z(t)$  is locally free on  $W \setminus \{c\}$ . Let  $T_t$  be the torsion part of  $R^1 p_* \mathcal{I}_Z(t)$ :  $T_t$  is concentrated in  $t$  (or it is even zero). Set  $F_t = (R^1 p_* \mathcal{I}_Z(t))/T_t$ . Since  $W$  is a smooth curve,  $F_t$  is locally free. Let  $W'$  be a neighborhood of  $c$  such that over  $W'$  all  $F_t$  are free: it exists since there is only a finite number of nonzero  $F_t$ 's. Fix a basis of  $F_t$  on  $W'$  for all  $t$ : this allows to define a morphism  $j: W' \rightarrow \mathcal{V}$  such that if  $a, b \in W' \setminus \{c\}$ , then  $j(a)$  is isomorphic to  $j(b)$ . Let us denote by  $j_t(a)$  the  $t$ th graded component of  $j(a)$ . Thus  $j(c)$  is in the closure of the orbit of  $j(b), b \in W' \setminus \{c\}$ . Again by base-change theorem, the natural maps

$$R^1 p_* \mathcal{I}_Z(t) \otimes k(c) \rightarrow H^1(\mathbf{P}^3, I_{p^{-1}(c)}(t))$$

are isomorphisms for every  $t$ .

Since tensor product is right exact,  $j_t(c) = F_t \otimes k(c)$  for every  $t$  is a quotient (as vector space) of  $H^1(\mathbf{P}^3, I_{p^{-1}(c)}(t))$ , and since these maps are natural and commute with the multiplication maps we get that  $j(c)$  is isomorphic, as graded module, to a quotient of  $\bigoplus_t H^1(\mathbf{P}^3, I_{p^{-1}(c)}(t))$ , that is to say to a quotient of the Hartshorne–Rao module of  $p^{-1}(c)$ .

Note that we can apply the base-change theorem since we are dealing with a family of curves with fixed speciality.

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