DEFICIENCY MODULES AND SPECIALIZATIONS

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Abstract. Given a family of curves in the projective space we study how their deficiency modules can change. This has a geometrical translation in the problem of determining how the liaison class of a flat family of curves can change. As a consequence, we show that in every liaison class there are curves which are specializations of arithmetically Cohen–Macaulay curves.

1. Introduction and preliminaries

The curves of the projective space $\mathbb{P}^3$ with fixed Hilbert polynomial are parameterized by the Hilbert scheme $H_{d,g}$. The cohomology dimensions $h^i(\mathbb{P}^3, I_{X_\alpha}(t))$, with $0 \leq i \leq 3$ and $t \in \mathbb{Z}$, when $X_\alpha$ varies in $H_{d,g}$, are upper semicontinuous and this is often used in order to distinguish between different irreducible components of $H_{d,g}$.

Here we investigate the possible variations of the module structures of the graded modules $\oplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, I_{X_\alpha}(t))$. These modules, called Hartshorne–Rao modules or deficiency modules, have a very important geometrical meaning, since they individuate the liaison class of $X_\alpha$.

In the second paragraph we prove that given any graded $k[x_0, x_1, x_2, x_3]$-module of finite length $A$ and a quotient $A/B$, there exist irreducible families of curves whose general member is in the liaison class individuated by $A/B$, and which specialize to a curve in the liaison class individuated by $A$. As a corollary we get that every liaison class contains a curve which is specialization of a flat family of arithmetically Cohen–Macaulay curves. These families have the property that the dimensions $h^1(X_\alpha, O_{X_\alpha}(t))$ don't depend on $\alpha$ (we call them families with fixed speciality). This "quotient" condition is the natural one, since in the third paragraph we show that in a family of curves with fixed speciality the Hartshorne–Rao module of the special curve has a quotient which is in the closure (in a suitable variety) of the isomorphism class of the Hartshorne–Rao module of the general curve. Roughly speaking, the only obstruction is semicontinuity.
If $k$ is an algebraically closed field, we set $S = k[x_0, x_1, x_2, x_3]$; with $P^3$ we denote the three-dimensional projective space over $k$; a curve will be a closed locally Cohen–Macaulay generically locally complete intersection one-dimensional subscheme of $P^3$.

We need some generalities about liaison of curves and Hartshorne–Rao modules: for these we refer to the paper of Rao [R].

In [BB] we defined a variety parametrizing all possible structures of graded $S$-module which are compatible with a given “graded” $k$-vector space structure. Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded $S$-module of finite length, and let $(m_1, \ldots, m_t)$ be a $t$-tuple of nonnegative integers such that $m_1 > 0$, $m_i > 0$. We say that $M$ is of type $(m_1, \ldots, m_t)$ if, up to shifting degrees, $M_i = 0$ if $i \leq 0$, $i > t$, and $\dim_k M_i = m_i$ if $1 \leq i \leq t$ (every degree of $M$ has a structure of $k$-vector space). Let now $V$ be the vector space $V = \bigoplus_{i=1}^t k^m_i = \bigoplus_{i=1}^t V_i$, and let us fix the canonical basis of $V$, and let $G = \{g \in S| \deg(g) = 1\}$.

Definition. $\mathcal{Y}' = \mathcal{Y}'_{P^3}(m_1, \ldots, m_t) = \{f = (f_1, \ldots, f_t) \in \bigoplus_{i=1}^{t-1} \text{Hom}(V_i \otimes G, V_{i+1})| g \in G, \forall \alpha \in V_i, \forall i, 1 \leq i \leq t-2, f_{i+1}[f_i(\alpha \otimes g) \otimes h] = f_{i+1}[f_i(\alpha \otimes g) \otimes g]\}$ is called the variety of module structures of finite length over $V$ of type $(m_1, \ldots, m_t)$.

Fix a graded $S$-module $M = \bigoplus_{i=1}^t M_i$ of type $(m_1, \ldots, m_t)$, and a basis $\mathcal{B}_i$ for the vector space $M_i$; then to $(M, \{\mathcal{B}_i\})$ is associated an element of $\mathcal{Y}'_{P^3}(m_1, \ldots, m_t)$ if we send each basis $\mathcal{B}_i$ to the canonical basis of $V_i$, since the multiplication is completely known if we know the vector space structure and the multiplication by elements of $G$. In this way we don’t identify isomorphic module structure, but every isomorphism class of module structures is a locally closed irreducible subset of $\mathcal{Y}'$. The structure and the properties of $\mathcal{Y}'$ are described in [BB].

2. Existence of specializations

In this paragraph we prove that, under a very natural algebraic hypothesis on the Hartshorne–Rao modules, there exist families of curves belonging to a given liaison class which specialize to curves of a second liaison class, and that this specialization is with fixed speciality.

Definition. A flat family $p: Z \to W$ of curves in $P^3$ (Z contained in $W \times P^3$, and $p$ induced by the projection) is said to have fixed speciality for all $t \in Z$ and all $a \in W, b \in W$ the following equality is verified:

$$h^1(p^{-1}(a), O_{p^{-1}(a)}(t)) = h^1(p^{-1}(b), O_{p^{-1}(b)}(t)).$$

Lemma 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves, $rk \mathcal{F} \leq rk \mathcal{G} - 2$, and

$$0 \to \mathcal{F} \to \mathcal{G}$$

be a morphism which drops rank in codimension $> 2$. Then there exist a direct sum of line bundles $\mathcal{P}$, with $rank \mathcal{P} = rank \mathcal{G} - rank \mathcal{F} - 1$ and a morphism
0 \rightarrow \mathcal{P} \xrightarrow{\sigma} \mathcal{G} \text{ such that } \\
(\varphi, \sigma) : \mathcal{I} \oplus \mathcal{P} \rightarrow \mathcal{G}

drops rank in codimension 2.

Proof. We have an exact sequence 

\[ 0 \rightarrow \mathcal{I} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{I} \rightarrow 0. \]

Take \( t >> 0 \) such that \( \mathcal{G}/\mathcal{I}(t) \) is globally generated and \( H^1(\mathbb{P}^3, \mathcal{I}(t)) = 0 \).

Take \( p = \text{rk} \mathcal{I} - \text{rk} \mathcal{I} - 1 \) general sections of \( \mathcal{G}/\mathcal{I}(t) \) and lift them to \( p \) sections of \( \mathcal{I}(t) \). In such a way we get a morphism \( 0 \rightarrow (O(-t))^p \rightarrow \mathcal{G} \) such that \((\varphi, \sigma)\) drops rank in codimension 2.

Proposition 2.2. Let \( A \) be a graded \( S \)-module of finite length, and \( A/B \) a quotient of \( A \); let \( \mathcal{L} \) and \( \mathcal{M} \) be the liaison classes of curves of \( \mathbb{P}^3 \) whose Hartshorne-Rao modules are isomorphic to \( A \) and \( A/B \) respectively. Then there exist infinitely many \((d, g)\) for which there is an irreducible family \( \{X_t\} \) of curves of degree \( d \) and genus \( g \) with fixed speciality whose general member \( X_t, t \neq t_0 \), is in the liaison class \( \mathcal{M} \), and \( X_{t_0} \in \mathcal{L} \).

Proof. We construct two locally free sheaves \( \mathcal{A} \) and \( \mathcal{B} \) whose dependency loci are curves in the liaison classes individuated by \( A \) and \( B \), respectively, simply following Rao's construction. To do this, we consider a set of generators (as graded \( S \)-module) of \( B \), and we extend it to a set of generators of \( A \); we go on in constructing free resolutions of \( B \) and \( A \) in this way and we consider the second syzygies modules of \( B \) and \( A \), let us call them \( B_2 \) and \( A_2 \). By sheafifying them we get two locally free sheaves \( \mathcal{A}' \) and \( \mathcal{B}' \) and, thanks to the functoriality of the sheafification, a morphism \( \gamma : \mathcal{B}' \rightarrow \mathcal{A}' \). Exactly as in [R], one gets that

\[ \bigoplus_i H^1(\mathbb{P}^3, \mathcal{B}'(t)) = B \quad \text{and} \quad \bigoplus_i H^1(\mathbb{P}^3, \mathcal{A}'(t)) = A, \]

\[ H^2(\mathbb{P}^3, \mathcal{B}'(t)) = 0 = H^2(\mathbb{P}^3, \mathcal{A}'(t)) \]

and the map induced by \( \gamma \) is exactly the inclusion \( B \rightarrow A \).

By adding a suitable sum of line bundles \( \mathcal{P} \) to \( \mathcal{A}' \), we can find morphisms

\[ \varphi = \gamma \oplus \sigma : \mathcal{B}' \rightarrow \mathcal{A}' \oplus \mathcal{P} \]

and

\[ \varphi_0 = 0 \oplus \tau : \mathcal{B}' \rightarrow \mathcal{A}' \oplus \mathcal{P} \]

which are injective morphisms of vector bundles. Let us call \( \mathcal{A}' \oplus \mathcal{P} = \mathcal{A} \).

Now we apply Lemma 2.1 to \( \varphi \) and to \( \varphi_0 \); there exist a direct sum of line bundles \( \mathcal{H} \) and morphisms \( \psi, \psi_0 \) such that

\[ \zeta : (\varphi, \psi) : \mathcal{B} \rightarrow \mathcal{A} \]

and

\[ \mu : (\varphi_0, \psi_0) : \mathcal{B} \rightarrow \mathcal{A} \]

drop rank in codimension 2, where \( \mathcal{B} = \mathcal{B}' \oplus \mathcal{H} \).
Morphisms having a two-codimensional dependency locus form an open non-
void set in $\mathrm{Hom}(\mathcal{B}, \mathcal{A})$; hence there is an open nonvoid subset of $k$ (the base
field) for which the morphism $\Lambda_t = t\zeta + (1 - t)\mu: \mathcal{B} \to \mathcal{A}$ drops rank in
codimension two.

Thus we have, for these $t$, exact sequences

$$0 \to \mathcal{B} \to \mathcal{A} \to I_{X_t}(r) \to 0$$

and from the long exact sequences associated to them we see that $H^2(P^3, I_{X_t}(p))$
does not depend on $t(H^2(P^3, \mathcal{A}(t)) = 0)$. This shows that the family $\{X_t\}$ has
fixed speciality.

Moreover, for $t \neq 0$ the map induced in cohomology by $\Lambda_t$ is the inclusion
$B \to A$; hence the Hartshorne–Rao module of $X_t$ is isomorphic to $A/B$ (since
$H^2(P^3, \mathcal{B}(t)) = 0$). For $t = 0$ the map induced in cohomology is zero, and
hence the Hartshorne–Rao module of $X_0$ is isomorphic to $A$.

In order to get infinite values of $(d, g)$ it is enough to use different twistings
of the line bundles which appear in the construction.

**Corollary 2.3.** Every liaison class contains a curve which is specialization of a
flat family of arithmetically Cohen–Macaulay curves.

**Proof.** Take $B = A$; the liaison class of curves with trivial Hartshorne–Rao
module is exactly the class of arithmetically Cohen–Macaulay curves.

**Example 2.4.** We follow step by step the proof of the proposition of this section
in order to produce an explicit example of this phenomenon.

Let $L_1$ be the liaison class of two skew lines; the elements of this class
have Hartshorne–Rao module concentrated in one degree, of dimension one.
The vector bundle that arises from Rao’s construction from this module is the
cotangent bundle $\Omega_{P^3}$. It is well known that there exists an exact sequence of
vector bundles

$$0 \to \Omega_{P^3} \to [\Omega_{P^3}(-1)]^4 \to \Omega_{P^3} \to 0.$$

Take as $\gamma$ (Proposition 2.2) the identity map $1: \Omega_{P^3} \to \Omega_{P^3}$. We have two
injective morphisms of vector bundles

$$\Phi = 1 \oplus 0: \Omega_{P^3} \to \Omega_{P^3} \oplus [\Omega_{P^3}(-1)]^4,$$

$$\Phi_0 = 0 \oplus \tau: \Omega_{P^3} \to \Omega_{P^3} \oplus [\Omega_{P^3}(-1)]^4,$$

whose cokernels are, respectively, $[\Omega_{P^3}(-1)]^4$ and $\Omega_{P^3} \oplus \Omega_{P^3}$.

Since both $[\Omega_{P^3}(-1)]^4(2)$ and $\Omega_{P^3} \oplus \Omega_{P^3}(2)$ are globally generated, and
$H^1(P^3, \Omega_{P^3}(2)) = 0$, we can find morphisms

$$\Psi, \Psi_0: [\Omega_{P^3}(-2)]^3 \to \Omega_{P^3} \oplus [\Omega_{P^3}(-1)]^4$$

such that

$$\zeta: (\Phi, \Psi): \Omega_{P^3} \oplus [\Omega_{P^3}(-2)]^3 \to \Omega_{P^3} \oplus [\Omega_{P^3}(-1)]^4,$$

$$\mu: (\Phi_0, \Psi_0): \Omega_{P^3} \oplus [\Omega_{P^3}(-2)]^3 \to \Omega_{P^3} \oplus [\Omega_{P^3}(-1)]^4,$$

drop rank in codimension 2, and hence the family of morphisms $\Lambda_t$. 
Therefore we have a family of exact sequences

\[ 0 \to \Omega_{P^3} \oplus [O_{P^3}(-2)]^3 \xrightarrow{\Lambda} \Omega_{P^3} \oplus [O_{P^3}(-1)]^4 \to I_{X_t}(2) \to 0 \]

(the twist of \( I_{X_t} \) is determined by the Chern classes), where \( X_t \) is a curve with \( \deg(X_t) = 6 \) and \( g(X_t) = 3 \) for every \( t \).

Note that \( X_t \) is arithmetically Cohen–Macaulay for \( t \neq 0 \), whilst \( X_0 \in L_1 \). Moreover,

\[ H^0(P^3, I_{X_t}(2)) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \neq 0. \end{cases} \]

Hence this is a family of hyperelliptic sextic curves (of genus 3), which specializes to a curve of bidegree \((2,4)\) lying on a quadric surface (directly linked to two skew lines).

### 3. The Hartshorne–Rao Module of a Specialization

In this section we go in the opposite direction: that is to say, we study how we can change the Hartshorne–Rao module in a flat family of curves. In particular, we concentrate on the case of a family with fixed speciality.

**Proposition 3.1.** Let \( p: Z \to W \) be a flat family of smooth curves in \( P^3 \), with \( \dim(W) > 0 \) and \( W \) irreducible, with fixed speciality. Assume the existence of \( c \in W \) such that for all \( a, b, \in W \setminus \{c\} \) the Hartshorne–Rao modules of \( p^{-1}(a) \) and \( p^{-1}(b) \) are isomorphic, and belong to the variety of module structures \( \mathcal{V} \). Then the Hartshorne–Rao module \( M \) of \( p^{-1}(c) \) has a quotient \( Q \in \mathcal{V} \) such that \( Q \) is in the closure in \( \mathcal{V} \) of the set of modules isomorphic to the Hartshorne–Rao module of \( p^{-1}(a) \).

**Proof.** Without losing generality we may assume \( W \) reduced, affine and of dimension 1. By base-change theorem ([M], Cor. 2 p. 50–51) for all \( t \in Z \)

\[ R^1 p \ast O_Z(t) \text{ (and } R^2 p \ast I_Z(t) \text{) are locally free and } R^1 p \ast I_Z(t) \text{ is locally free on } W \setminus \{c\} \text{.} \]

Let \( T_t \) be the torsion part of \( R^1 p \ast I_Z(t) \): \( T_t \) is concentrated in \( t \) (or it is even zero). Set \( F_t = (R^1 p \ast I_Z(t))/T_t \). Since \( W \) is a smooth curve, \( F_t \) is locally free. Let \( W' \) be a neighborhood of \( c \) such that over \( W' \) all \( F_t \) are free: it exists since there is only a finite number of nonzero \( F_t \)'s. Fix a basis of \( F_t \) on \( W' \) for all \( t \); this allows to define a morphism \( j: W' \to \mathcal{V} \) such that if \( a, b, \in W' \setminus \{c\} \), then \( j(a) \) is isomorphic to \( j(b) \). Let us denote by \( j_t(a) \) the \( t \)th graded component of \( j(a) \). Thus \( j(c) \) is in the closure of the orbit of \( j(b), b, \in W' \setminus \{c\} \). Again by base-change theorem, the natural maps

\[ R^1 p \ast I_Z(t) \otimes k(c) \to H^1(P^3, I_{p^{-1}(c)}(t)) \]

are isomorphisms for every \( t \).
Since tensor product is right exact, \( j_t(c) = F_t \otimes k(c) \) for every \( t \) is a quotient (as vector space) of \( H^1(P^3, I_{p^{-1}(c)}(t)) \), and since these maps are natural and commute with the multiplication maps we get that \( j(c) \) is isomorphic, as graded module, to a quotient of \( \bigoplus H^1(P^3, I_{p^{-1}(c)}(t)) \), that is to say to a quotient of the Hartshorne–Rao module of \( p^{-1}(c) \).

Note that we can apply the base-change theorem since we are dealing with a family of curves with fixed speciality.

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References


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