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A SIMPLE EXAMPLE OF A NORMAL OPERATOR T ON A BANACH SPACE SUCH THAT $r(T) < \|T\|$

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ABSTRACT. Bonsall and Duncan's book ([1, Theorem 25.6]) exhibits a normal element $h + ik$ of a Banach algebra such that $r(h + ik) < \|h + ik\|$. In this paper, we will give a simpler example of a normal operator T on a Banach space such that $r(T) < \|T\|$.

1. **Example.** Let

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \qquad \text{where } \omega = \tfrac{1}{2}(-1 + \sqrt{3}i).$$

Consider the derivation D_T on the Banach space $B(\mathbf{C}^3)$ (set of all 3 by 3 matrices with the operator norm): $D_T(X) := TX - XT$. Then since T is normal, D_T is a normal operator on $B(\mathbf{C}^3)$ ([2, Theorem 2.7]). Rosenblum's result then shows that

$$\sigma(D_T) = \sigma(T) - \sigma(T) = \{0, \pm(1-\omega), \pm(1-\omega^2), \pm(\omega - \omega^2)\}.$$

Therefore, $r(D_T) = \sqrt{3}$, where $r(D_T)$ is the spectral radius of D_T. Since T is normal, it follows, by Corollary 1.1 in [4], that $\|D_T\| = 2 \cdot R_T$, where R_T is the radius of the smallest disk containing the spectrum of T. Hence, $\|D_T\| = 2$.

That is $r(D_T) = \sqrt{3} < 2 = \|D_T\|$.

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