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COMPLEMENTATION OF JORDAN TRIPLES IN VON NEUMANN ALGEBRAS

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ABSTRACT. We show that the predual of a JBW^*-triple is complemented in the predual of a von Neumann algebra. Hence a quotientof a JB^*-triple does not contain l_1 if and only if its dual enjoys the Radon–Nikodym property. We also show that a JB^*-triple either contains c_0 or is reflexive.

1. INTRODUCTION

We show that every JBW^*-triple is isomorphic as a Banach space to a complemented subspace of a von Neumann algebra. Hence many Banach space properties of operator algebras do indeed pass on to Jordan triples. In fact, we prove that the predual of a JBW^*-triple J is complemented in the predual of a von Neumann algebra and J is even isomorphic as a Jordan triple to a 1-complemented subtriple of a von Neumann algebra if (and only if) J does not contain the exceptional Cartan factors C^5 and C^6. Moreover, if A is a JBW^*-algebra not containing C^6, then it is the range of a positive contractive projection on a von Neumann algebra. This is an extension of a result of Effros and Strømer [5]. One application is that a quotient X of a JB^*-triple does not contain l_1 if and only if its dual X^* has the Radon–Nikodym property. In conjunction with this result, we also prove that a JB^*-triple either contains c_0 or is reflexive. So the Krein–Milman property and the Radon–Nikodym property are equivalent in JB^*-triples.

Briefly JB^*-triples are all those Banach spaces whose open unit balls are bounded symmetric domains [15]. These include C^*-algebras and the larger class of JB^*-algebras (Jordan C^*-algebras) as well as Hilbert spaces. A JB^*-*triple* is a complex Banach space J equipped with a triple product $\{\cdots\}\colon J \times J \times J \to J$ such that

(i) $\{xyz\}$ is bilinear and symmetric in x and z, but antilinear in y,
(ii) $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{aby\}z\} + \{xy\{abz\}\}$,

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(iii) the left multiplication $\{xx.\}: J \to J$ is Hermitian and has nonnegative spectrum,

(iv) $\|\{xxx\}\| = \|x\|^3$.

For instance, the triple product $\{xyz\} = 2^{-1}(xy^*z + zy^*x)$ in a C^*-algebra satisfies all these conditions. If, moreover, J has a (necessarily unique) predual J_*, then it is called a *JBW^*-triple* and the w^*-topology on J refers to the topology $\sigma(J, J_*)$. The second dual of a JB^*-triple is a JBW^*-triple. Other examples are the so-called Cartan factors C^k $(k = 1, 2, \dots, 6)$ where C^4 is a spin factor, C^5 is the (finite-dimensional) exceptional Cartan factor consisting of 1×2 matrices over the complex Cayley numbers $\mathbf{O}$ and C^6 consists of 3×3 Hermitian matrices over $\mathbf{O}$. In fact, C^3, C^4, C^6 are JBW^*-algebras. An element e in J is a *tripotent* if $\{eee\} = e$. It induces a Pierce decomposition $J = J_1(e) \oplus J_{1/2}(e) \oplus J_0(e)$ as an l_∞-sum, where $J_k(e) = \{x \in J: \{eex\} = kx\}$ $(k = 0, \frac{1}{2}, 1)$ is the eigenspace of $\{ee.\}$, called the *Pierce k-space* of e. Also $J_k(e)$ is the range of the *Pierce k-projection* $P_k(e): J \to J$ where

$$P_1(e)(z) = \{e\{eze\}e\}$$
$$P_{1/2}(e)(z) = 2(\{eez\} - P_1(e)(z))$$
$$P_0(e)(z) = z - 2\{eez\} + P_1(e)(z).$$

Moreover, if J is a JBW^*-triple, then the Pierce projections are w^*-continuous. A tripotent e is *complete* if $J_0(e) = \{0\}$. The complete tripotents are precisely the extreme points of the closed unit ball of J. Tripotents abound in JBW^*-triples by the Krein–Milman theorem and they are just the partial isometries in C^*-algebras. For further properties of JB^*-triples, we refer to [10], [11], [12], [15], [18].

2. Complementation in von Neumann algebras

Abusing the language slightly, we say that a Banach space X is *complemented* in the space Y if X is isometric to a complemented subspace Z of Y, i.e. there is a bounded projection $P: Y \to Z$. If moreover, $\|P\| = 1$, we say that X is *1-complemented* in Y. We note that the predual of a Cartan factor is 1-complemented in the predual of a JBW^*-algebra. Indeed, the *rectangular Cartan factor* $C^1 = L(H, K)$ consists of bounded operators between Hilbert spaces H and K. So the predual $L(H, K)_*$ of C^1 is 1-complemented in the predual of the von Neumann algebra $L(H \oplus K)$. The *symplectic Cartan factor* C_n^2 has the form $\{z \in L(H): z = -jz^*j\}$ where $j: H \to H$ is a conjugate linear isometric involution and $n = \dim H$. So its predual equals $\{t \in L(H)_*: t = -jt^*j\}$ which is 1-complemented in $L(H)_*$ where $L(H)_*$ identifies with the trace-class operators on H. The *Hermitian Cartan factor* C^3 has the form $\{t \in L(H): t = jt^*j\}$ and its predual is 1-complemented in $L(H)_*$. Also, if e is a minimal tripotent in C^6, then the Pierce projection $P_{1/2}(e): C^6 \to C^6$ has range C^5 and so C_*^5 is 1-complemented in C_*^6.

Lemma 1. *Let J be a JBW^*-triple. Then its predual J_* is 1-complemented in the predual of a JBW^*-algebra.*

Proof. By [10], [12], J admits the following l_∞-sum

$$J = \oplus_\alpha C(\Omega_\alpha, C^\alpha) \oplus J^7 \oplus J^8$$

where $C(\Omega_\alpha, C^\alpha)$ is the space of continuous functions from a hyperstonean space Ω_α to a Cartan factor C^α, J^7 is a continuous JW^*-algebra and J^8 is a w^*-closed right ideal of a von Neumann algebra M. The predual J^8_* is clearly 1-complemented in M_*. The predual of $C(\Omega_\alpha, C^\alpha)$ is just the projective tensor product $L_1(\Sigma_\alpha)\hat{\otimes}C^\alpha_*$ of an L_1-space and the predual of the Cartan factor C^α. Now C^α_* is 1-complemented in the predual of some JBW^*-algebras A^α, it follows that $L_1(\Sigma_\alpha)\hat{\otimes}C^\alpha_*$ is 1-complemented in $L_1(\Sigma_\alpha)\hat{\otimes}A^\alpha_*$ which is the predual of the JBW^*-algebra $L_\infty(\Sigma_\alpha, A^\alpha)$. Thus the l_1-sum $J_* = \oplus_\alpha(L_1(\Sigma_\alpha)\hat{\otimes}C^\alpha_*) \oplus J^7_* \oplus J^8_*$ is 1-complemented in the l_1-sum $\oplus_\alpha L_\infty(\Sigma_\alpha, A^\alpha)_* \oplus J^7_* \oplus M_*$.

Moreover, by [5, Lemma 2.3], every spin factor C^4 is the range of a unital positive contractive projection on the von Neumann algebra generated by C^4. By taking the bidual of the projection and noting that C^4 is reflexive, we see that C^4 embeds as a Jordan subalgebra in a von Neumann algebra M and the predual of C^4 is 1-complemented in M_*. Although C^5 and C^6 are clearly complemented in von Neumann algebras by finite-dimensionality, they cannot be 1-complemented because the range of a contractive projection on a von Neumann algebra has the structure of a J^*-algebra [6; Théorème 2]. We thus reach the following conclusion.

Theorem 2. *Let J be a JBW^*-triple. Then its predual J_* is complemented in the predual of a von Neumann algebra. Moreover, J_* is 1-complemented in the predual of a von Neumann algebra if and only if J does not contain C^5 and C^6.*

Proof. As in Lemma 1, we write $J = \oplus_\alpha C(\Omega_\alpha, C^\alpha) \oplus J^7 \oplus J^8$ where, by [14; 1.20], $J^7 = \{a \in M : \Theta(a) = a\}$ where $\Theta: M \to M$ is a w^*-continuous *-antiautomorphism of period 2 on a von Neumann algebra M. So J^7 is 1-complemented in M by the bicontractive projection $2^{-1}(Id + \Theta)$. Same arguments as before complete the proof.

The Cartan factors C^k is isomorphic as Jordan subtriple of a von Neumann algebra if and only if $k \neq 5, 6$. So we have the following result by taking direct sums as before.

Corollary 3. *Let J be a JBW^*-triple. Then J is isomorphic as a Banach space to a complemented subspace of a von Neumann algebra. Moreover, J is isomorphic as a Jordan triple to a 1-complemented subtriple of a von Neumann algebra if and only if it does not contain C^5 and C^6.*

Remark 4. In the case of a JBW^*-algebra A without the summand $C(\Omega, C^6)$, it can be embedded as a Jordan subalgebra of a von Neumann algebra M and is

1-complemented by a w^*-continuous positive contractive projection on M. Indeed, we can write $A = \oplus_\alpha C(\Omega_\alpha, C^\alpha) \oplus B$ where C^α are spin factors and B is a JW^*-algebra 1-complemented in a von Neumann algebra by a w^*-continuous bicontractive projection E [9; 5.3.5, 6.3.14, 7.3.5]. As remarked before, C^α is the range of a w^*-continuous contractive projection $P_\alpha : M \to M$ on a von Neumann algebra M whose predual Q_α has C^α_* for range. Now P_α lifts to a projection

$$E_\alpha : C(\Omega_\alpha, M^\alpha) \to C(\Omega_\alpha, C^\alpha) \text{ with predual}$$
$$Q_\alpha \otimes 1 : L_1(\Sigma_\alpha)\hat{\otimes} M^\alpha_* \to L_1(\Sigma_\alpha)\hat{\otimes} C^\alpha_* .$$

Therefore A is the range of the l_∞-sum of the projections E_α and E which is w^*-contiunuous and contractive, hence positive by [9; 4.4.13].

3. Banach space properties

It is well known that a Banach space does not contain l_1 if and only if its dual has the weak Radon–Nikodym property [14, 16]. Applying Theorem 2, we have the following sharper result for JB^*-triples which is an improvement of [2; Theorem 2], see also [17].

Theorem 5. *Let X be a quotient of a JB^*-triple J. Then X does not contain l_1 if and only if X^* has the Radon–Nikodym property.*

Proof. First X^* is isometric to a subspace of J^* and by Theorem 2, J^* is complemented in the predual M_* of a von Neumann algebra M. But M_* is complemented in $(M_*)^{**}$ and every separable subspace of M_* is contained in a separable complemented subspace of M_* (see [7; Appendix]), therefore the closed unit ball of X^* has the Radon–Nikodym property if and only if it is strongly regular in the sense of [7] which in turn is equivalent to $X \not\supset l_1$ by [7; VIII.4 and VIII.18].

Finally we prove that every JB^*-triple either contains c_0 or is reflexive.

Theorem 6. *Let J be a JB^*-triple. Then the following conditions are equivalent:*

(i) *J has the Krein–Milman property,*
(ii) *J does not contain c_0,*
(iii) *J is reflexive,*
(iv) *J has the Radon–Nikodym property.*

Proof. (ii) $\Rightarrow$ (iii). Take any x in J. Then the JB^*-triple J_x generated by x is an Abelian C^*-algebra [15]. Now $J_x \not\supset c_0$ implies J_x is finite dimensional since the identity map on J_x is weakly compact [1] and so J_x contains mutually orthogonal tripotents. Let $\{e_1, \dots, e_n\}$ be a maximal family of orthogonal tripotents in J. As in [3; p. 186], we have $n < \infty$ for otherwise J would contain a copy of c_0. Let $e = e_1 + \cdots + e_n$. We show that e is a complete tripotent in J. Indeed, if $z \in J_0(e)$ and $z \neq 0$, then by considering $J_z \subset J_0(e)$, there is a nontrivial tripotent $u \in J_z \subset J_0(e)$. By orthogonality, we have $e_k \in J_1(e)$ which implies $J_0(e) \subset J_0(e_k)$ (cf. [11; 1.14]) and hence $u \in J_0(e_k)$, that is, u is orthogonal to e_k for $k = 1, \ldots, n$. This contradicts maximality. So $J_0(e) = \{0\}$. Recall that the Pierce k-space $J_k^{**}(e)$ in J^{**} is the range of the Pierce projection $P_k(e)\colon J^{**} \to J^{**}$ and as $e \in J$, we also have $J_k(e) = P_k(e)J$. But $P_k(e)$ is w^*-continuous, it follows that $J_k(e)$ is w^*-dense in $J_k^{**}(e)$. In particular, $J_0^{**}(e) = \{0\}$ and e is a complete tripotent in J^{**}. Now $J_1(e) \not\supset c_0$ implies that $J_1(e)$ is relfexive by [4; Theorem 8]. The predual of $J_1^{**}(e)$ is $J^*/J_1^{**}(e)^0$ where $J_1^{**}(e)^0$ is the polar in J^* [11; 3.7]. As $J_1(e)$ is w^*-dense in $J_1^{**}(e)$, $J_1^{**}(e)^0$ coincides with the polar $J_1(e)^0$ of $J_1(e)$ in J^*. Thus $J_1(e)^* = J^*/J_1(e)^0$ is the predual of $J_1^{**}(e)$ and hence $J_1(e) = J_1(e)^{**} = J_1^{**}(e)$. In other words, $J_1(e)$ is the Pierce 1-space of e in J^{**}.

Now $J_1(e)$ is a finite l_∞-sum $A_1 \oplus \cdots \oplus A_n$ of relfexive Cartan factors [4; p. 457]. Hence, by [10; 1.12], we have an l_∞-sum $J^{**} = U_1 \oplus \cdots \oplus U_n$ of w^*-closed triple ideals with $A_j = U_j \cap J_1(e)$. If we write $e = u_1 + \cdots + u_n$, then u_j is a complete tirpotent in U_j such that its Pierce 1-space $(U_j)_1(u_j)$ is just A_j. To complete the proof, we show that J^{**} is reflexive. It suffices to show that each U_j is reflexive. First U_j is a Cartan factor by [11; 4.6]. Note that the Cartan factors C^4, C^5, and C^6 are reflexive. If U_j is a rectangular Cartan factor, then by [10; 5.5], $A_j = (U_j)_1(u_j)$ contains a complete rectangular grid (e_{lk}) $(l, k \in I)$ with $u = \sum_{l \in I} e_{ll}$. Also $U_j = L(H, K)$ with $\dim K$ equals to card I. As $A_j \not\supset c_0$, A_j cannot contain infinitely many mutually orthogonal tripotents and hence card $I < \infty$. So U_j is reflexive in this case. If U_j is symplectic, then by [10; 6.1] $A_j = C_{2n}^2$ with $n < \infty$ and U_j is either C_{2n}^2 or C_{2n+1}^2, and hence it is reflexive. Finally, if U_j is hermitian type, i.e. of type C^3, then by [10; 7.1], u_j is unitary which means $U_j = A_j$ and so U_j is reflexive.

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